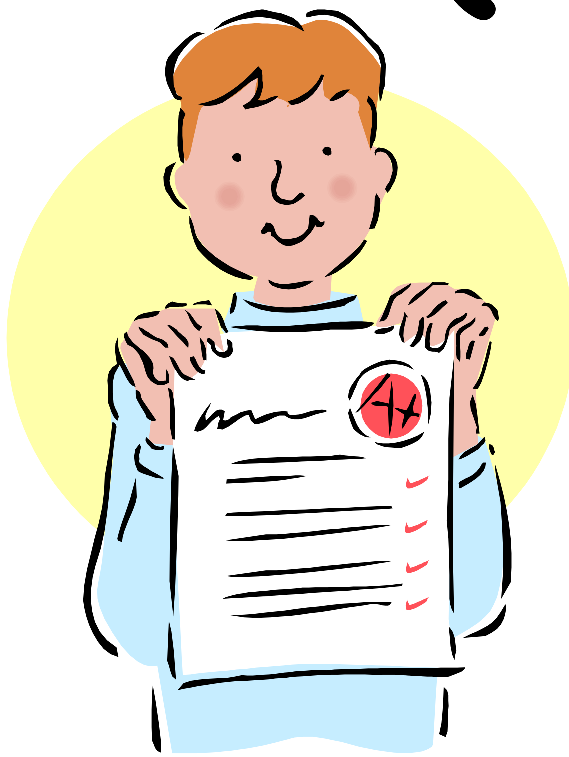
Core 3 (A2)



Practice Examination Questions

Trigonometric Identities and Equations

Trigonometry	I know what secant; cosecant and cotangent graphs look like and can identify appropriate restricted domains.	
	I know and can use the relationship between secant and cosine.	
	I know and can use the relationship between cosecant and sine.	
	I know and can use the relationship between cotangent and tangent.	
	I know how arcsin, arccos and arctan relate to sine, cosine and tangent.	
	I understand the graphs of arcsin, arcos and arctan.	
	I know and can use the identity: $\sin^2\theta + \cos^2\theta = 1$	
	I know and can use the identity: $\sec^2\theta = 1 + \tan^2\theta$.	
	I know and can use the identity: $\text{cosec}^2\theta = 1 + \cot^2\theta$.	
	I can work with angles in both degrees and radians.	
	I know and can use $\sin(A\pm B)$ and related $\sin(2A)$.	
	I know and can use $\cos(A\pm B)$ and related $\cos(2A)$.	
	I know and can use $\tan(A\pm B)$ and related $\tan(2A)$.	

1. (*a*) Given that $\sin^2\theta + \cos^2\theta \equiv 1$, show that $1 + \tan^2\theta \equiv \sec^2\theta$.
(2)

(*b*) Solve, for $0 \leq \theta < 360°$, the equation

$$2\tan^2\theta + \sec\theta = 1,$$

giving your answers to 1 decimal place.
(6)

2. (*a*) By writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$
(5)

(*b*) Given that $\sin\theta = \frac{\sqrt{3}}{4}$, find the exact value of $\sin 3\theta$.
(2)

3. (*a*) Using $\sin^2\theta + \cos^2\theta \equiv 1$, show that the $\operatorname{cosec}^2\theta - \cot^2\theta \equiv 1$.
(2)

(*b*) Hence, or otherwise, prove that

$$\operatorname{cosec}^4\theta - \cot^4\theta \equiv \operatorname{cosec}^2\theta + \cot^2\theta.$$
(2)

(*c*) Solve, for $90° < \theta < 180°$,

$$\operatorname{cosec}^4\theta - \cot^4\theta = 2 - \cot\theta.$$
(6)

4. (*a*) Given that $\sin^2\theta + \cos^2\theta \equiv 1$, show that $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$.
(2)

(*b*) Solve, for $0 \leq \theta < 180°$, the equation

$$2\cot^2\theta - 9\operatorname{cosec}\theta = 3,$$

giving your answers to 1 decimal place.
(6)

5. (*a*) Show that

(i) $\dfrac{\cos 2x}{\cos x + \sin x} \equiv \cos x - \sin x, \quad x \neq (n - \tfrac{1}{4})\pi,\; n \in \mathbb{Z},$

(2)

(ii) $\tfrac{1}{2}(\cos 2x - \sin 2x) \equiv \cos^2 x - \cos x \sin x - \tfrac{1}{2}$.

(3)

(*b*) Hence, or otherwise, show that the equation

$$\cos\theta\left(\frac{\cos 2\theta}{\cos\theta + \sin\theta}\right) = \frac{1}{2}$$

can be written as

$$\sin 2\theta = \cos 2\theta.$$

(3)

(*c*) Solve, for $0 \le \theta < 2\pi$,

$$\sin 2\theta = \cos 2\theta,$$

giving your answers in terms of π.

(4)

6. (*a*) Given that $\cos A = \tfrac{3}{4}$, where $270° < A < 360°$, find the exact value of $\sin 2A$.

(5)

(*b*) (i) Show that $\cos\left(2x + \dfrac{\pi}{3}\right) + \cos\left(2x - \dfrac{\pi}{3}\right) \equiv \cos 2x$.

(3)

Given that

$$y = 3\sin^2 x + \cos\left(2x + \frac{\pi}{3}\right) + \cos\left(2x - \frac{\pi}{3}\right),$$

(ii) show that $\dfrac{dy}{dx} = \sin 2x$.

(4)

7. (i) Prove that

$$\sec^2 x - \operatorname{cosec}^2 x \equiv \tan^2 x - \cot^2 x.$$

(3)

(ii) Given that

$$y = \arccos x, \quad -1 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq \pi,$$

(*a*) express arcsin x in terms of y.

(2)

(*b*) Hence evaluate arccos x + arcsin x. Give your answer in terms of π.

(1)

8. (i) Without using a calculator, find the exact value of

$$(\sin 22.5° + \cos 22.5°)^2.$$

You must show each stage of your working.

(5)

(ii) (*a*) Show that $\cos 2\theta + \sin \theta = 1$ may be written in the form

$$k \sin^2 \theta - \sin \theta = 0, \text{ stating the value of } k.$$

(2)

(*b*) Hence solve, for $0 \leq \theta < 360°$, the equation

$$\cos 2\theta + \sin \theta = 1.$$

(4)

9. (*a*) Prove that

$$\frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \tan \theta, \qquad \theta \neq 90n°, \quad n \in \mathbb{Z}.$$

(4)

(*b*) Hence, or otherwise,

(i) show that $\tan 15° = 2 - \sqrt{3}$,

(3)

(ii) solve, for $0 < x < 360°$,

$$\operatorname{cosec} 4x - \cot 4x = 1.$$

(5)

Differentiation

Differentiation	I know and can use the differential of e^x.	
	I know and can use the differential of ln(x).	
	I know and can use the differential of sin(x).	
	I know and can use the differential of cos(x).	
	I know and can use the differential of tan(x).	
	I can use the chain rule to differentiate composite functions.	
	I can use the product rule to differentiate products.	
	I can use the quotient rule to differentiate fractions.	
	I can derive and use the differential of cosec(x).	
	I can derive and use the differential of sec(x).	
	I can derive and use the differential of cot(x).	
	I understand that $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$ when working with eg. $\frac{dy}{dx}$ for $x = \sin 3y$.	

1. (*a*) Differentiate with respect to x

(i) $3 \sin^2 x + \sec 2x$,

(3)

(ii) $\{x + \ln(2x)\}^3$.

(3)

Given that $y = \dfrac{5x^2 - 10x + 9}{(x-1)^2}$, $x \neq 1$,

(*b*) show that $\dfrac{dy}{dx} = -\dfrac{8}{(x-1)^3}$.

(6)

2. Differentiate, with respect to x,

(a) $e^{3x} + \ln 2x$,

(3)

(*b*) $(5 + x^2)^{\frac{3}{2}}$.

(3)

3. (*a*) Differentiate with respect to x

(i) $x^2 e^{3x+2}$,

(4)

(ii) $\dfrac{\cos(2x^3)}{3x}$.

(4)

(*b*) Given that $x = 4 \sin(2y + 6)$, find $\dfrac{dy}{dx}$ in terms of x.

(5)

4. The point P lies on the curve with equation $y = \ln\left(\frac{1}{3}x\right)$. The x-coordinate of P is 3.

Find an equation of the normal to the curve at the point P in the form $y = ax + b$, where a and b are constants.

(5)

4. The curve C has equation $x = 2 \sin y$.

(a) Show that the point $P\left(\sqrt{2}, \frac{\pi}{4}\right)$ lies on C.

(1)

(b) Show that $\frac{dy}{dx} = \frac{1}{\sqrt{2}}$ at P.

(4)

(c) Find an equation of the normal to C at P. Give your answer in the form $y = mx + c$, where m and c are exact constants.

(4)

5. (i) The curve C has equation $y = \frac{x}{9 + x^2}$.

Use calculus to find the coordinates of the turning points of C.

(6)

(ii) Given that $y = (1 + e^{2x})^{\frac{3}{2}}$, find the value of $\frac{dy}{dx}$ at $x = \frac{1}{2}\ln 3$.

(5)

6. A curve C has equation

$$y = e^{2x} \tan x, \quad x \neq (2n + 1)\frac{\pi}{2}.$$

(*a*) Show that the turning points on C occur where $\tan x = -1$.

(6)

(*b*) Find an equation of the tangent to C at the point where $x = 0$.

(2)

7. (*a*) Differentiate with respect to x,

(i) $e^{3x}(\sin x + 2\cos x)$,

(3)

(ii) $x^3 \ln(5x + 2)$.

(3)

Given that $y = \dfrac{3x^2 + 6x - 7}{(x+1)^2}$, $x \neq -1$,

(*b*) show that $\dfrac{dy}{dx} = \dfrac{20}{(x+1)^3}$.

(5)

(*c*) Hence find $\dfrac{d^2y}{dx^2}$ and the real values of x for which $\dfrac{d^2y}{dx^2} = -\dfrac{15}{4}$.

(3)

8. The curve C has equation

$$y = (2x - 3)^5$$

The point P lies on C and has coordinates $(w, -32)$.

Find

(*a*) the value of w,

(2)

(*b*) the equation of the tangent to C at the point P in the form $y = mx + c$, where m and c are constants.

(5)

9. (i) Differentiate with respect to x

(*a*) $y = x^3 \ln 2x$,

(*b*) $y = (x + \sin 2x)^3$.

(6)

Given that $x = \cot y$,

(ii) show that $\dfrac{dy}{dx} = \dfrac{-1}{1 + x^2}$.

(5)

10. Differentiate with respect to x

(*a*) $\ln (x^2 + 3x + 5)$,

(2)

(*b*) $\dfrac{\cos x}{x^2}$.

(3)

Algebraic Fractions

Algebraic Fractions	I can simplify algebraic fractions by factorising and cancelling.	
	I can perform addition and subtraction with algebraic fractions.	

Mr A Slack

1. Express

$$\frac{2x^2+3x}{(2x+3)(x-2)} - \frac{6}{x^2-x-2}$$

as a single fraction in its simplest form.

(7)

2. (*a*) Simplify $\frac{3x^2-x-2}{x^2-1}$.

(3)

(*b*) Hence, or otherwise, express $\frac{3x^2-x-2}{x^2-1} - \frac{1}{x(x+1)}$ as a single fraction in its simplest form.

(3)

3. Given that

$$\frac{2x^4-3x^2+x+1}{(x^2-1)} \equiv (ax^2+bx+c) + \frac{dx+e}{(x^2-1)},$$

find the values of the constants a, b, c, d and e.

(4)

4.

$$f(x) = 1 - \frac{3}{x+2} + \frac{3}{(x+2)^2}, \quad x \neq -2.$$

(*a*) Show that $f(x) = \frac{x^2+x+1}{(x+2)^2}$, $x \neq -2$.

(4)

(*b*) Show that $x^2 + x + 1 > 0$ for all values of x.

(3)

(*c*) Show that $f(x) > 0$ for all values of x, $x \neq -2$.

(1)

5. $$\text{h}(x) = \frac{2}{x+2} + \frac{4}{x^2+5} - \frac{18}{(x^2+5)(x+2)}, \quad x \geq 0.$$

(*a*) Show that $\text{h}(x) = \dfrac{2x}{x^2+5}$.

(4)

(*b*) Hence, or otherwise, find $\text{h}'(x)$ in its simplest form.

(3)

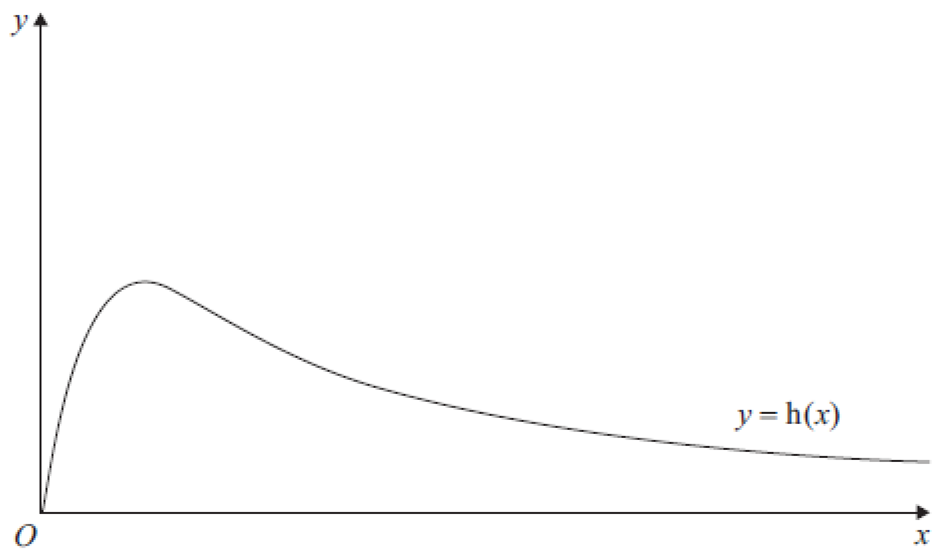


Figure 2

Figure 2 shows a graph of the curve with equation $y = \text{h}(x)$.

(*c*) Calculate the range of $\text{h}(x)$.

(5)

6. $$\text{f}(x) = \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{x^2-9}, \quad x \neq \pm 3, \ x \neq -\frac{1}{2}.$$

(*a*) Show that

$$\text{f}(x) = \frac{5}{(2x+1)(x+3)}.$$

(5)

The curve C has equation $y = \text{f}(x)$. The point $P\left(-1, -\dfrac{5}{2}\right)$ lies on C.

(*b*) Find an equation of the normal to C at P.

(8)

Functions

Algebra and Functions	I know the definition of a function and how it may be notated.	
	I know the meaning of 'one-to-one' and 'many-to-one' functions.	
	I understand and can use the "domain" of a function.	
	I understand and can use the "range" of a function.	
	I can work with composite functions.	
	I can find the inverse of a function.	
	I can draw the graph of an inverse function.	

1. The function f is defined by

$$f: x \mapsto \frac{5x+1}{x^2+x-2} - \frac{3}{x+2}, \quad x > 1.$$

(*a*) Show that $f(x) = \dfrac{2}{x-1}, \quad x > 1.$

(4)

(*b*) Find $f^{-1}(x)$.

(3)

The function g is defined by

$$g: x \mapsto x^2 + 5, \quad x \in \mathbb{R}.$$

(*b*) Solve $fg(x) = \frac{1}{4}$.

(3)

2. The functions f and g are defined by

$$f: x \mapsto 2x + \ln 2, \qquad x \in \mathbb{R},$$

$$g: x \mapsto e^{2x}, \qquad x \in \mathbb{R}.$$

(*a*) Prove that the composite function gf is

$$gf: x \mapsto 4e^{4x}, \qquad x \in \mathbb{R}.$$

(4)

(*b*) Sketch the curve with equation $y = gf(x)$, and show the coordinates of the point where the curve cuts the y-axis.

(1)

(*c*) Write down the range of gf.

(1)

(*d*) Find the value of x for which $\dfrac{d}{dx}[gf(x)] = 3$, giving your answer to 3 significant figures.

(4)

3. The function f is defined by

$$f : x \mapsto \ln (4 - 2x), \quad x < 2 \text{ and } x \in \mathbb{R}.$$

(*a*) Show that the inverse function of f is defined by

$$f^{-1} : x \mapsto 2 - \tfrac{1}{2} e^{x}$$

and write down the domain of f^{-1}.

(4)

(*b*) Write down the range of f^{-1}.

(1)

(*c*) Sketch the graph of $y = f^{-1}(x)$. State the coordinates of the points of intersection with the x and y axes.

(4)

The graph of $y = x + 2$ crosses the graph of $y = f^{-1}(x)$ at $x = k$.

The iterative formula

$$x_{n+1} = -\tfrac{1}{2} e^{x_n}, \quad x_0 = -0.3,$$

is used to find an approximate value for k.

(*d*) Calculate the values of x_1 and x_2, giving your answer to 4 decimal places.

(2)

(*e*) Find the values of k to 3 decimal places.

(2)

4. The functions f and g are defined by

$$\text{f} : x \mapsto 1 - 2x^3, \quad x \in \mathbb{R}.$$

$$\text{g} : x \mapsto \frac{3}{x} - 4, \quad x > 0, \ x \in \mathbb{R}.$$

(*a*) Find the inverse function f^{-1}.

(2)

(*b*) Show that the composite function gf is

$$\text{gf} : x \mapsto \frac{8x^3 - 1}{1 - 2x^3}.$$

(4)

(*c*) Solve gf (x) = 0.

(2)

(*d*) Use calculus to find the coordinates of the stationary point on the graph of $y = \text{gf}(x)$.

(5)

5. The function f is defined by

$$\text{f}: x \mapsto \frac{2(x-1)}{x^2 - 2x - 3} - \frac{1}{x-3}, \quad x > 3.$$

(*a*) Show that $\text{f}(x) = \dfrac{1}{x+1}, \quad x > 3$.

(4)

(*b*) Find the range of f.

(2)

(*c*) Find $\text{f}^{-1}(x)$. State the domain of this inverse function.

(3)

The function g is defined by

$$\text{g}: x \mapsto 2x^2 - 3, \quad x \in \mathbb{R}.$$

(*d*) Solve $\text{fg}(x) = \dfrac{1}{8}$.

(3)

6. The function f is defined by

$$f : x \mapsto 4 - \ln (x + 2), \quad x \in \mathbb{R}, \quad x \geq -1.$$

(*a*) Find $f^{-1}(x)$.

(3)

(*b*) Find the domain of f^{-1}.

(1)

The function g is defined by

$$g : x \mapsto e^{x^2} - 2, \quad x \in \mathbb{R}.$$

(*c*) Find fg(x), giving your answer in its simplest form.

(3)

(*d*) Find the range of fg.

(1)

Iteration

Numerical Methods	I can identify the location of roots of $f(x)=0$ by considering a change of sign of $f(x)$.	
	I can find an approximate solution to an equation using simple iterative methods including relations of the form $x_{n+1} = f(x_n)$.	

1. $$f(x) = 3e^x - \tfrac{1}{2}\ln x - 2, \quad x > 0.$$

(*a*) Differentiate to find $f'(x)$.

(3)

The curve with equation $y = f(x)$ has a turning point at P. The x-coordinate of P is α.

(*b*) Show that $\alpha = \tfrac{1}{6}e^{-\alpha}$.

(2)

The iterative formula

$$x_{n+1} = \tfrac{1}{6}e^{-x_n}, \quad x_0 = 1,$$

is used to find an approximate value for α.

(*c*) Calculate the values of x_1, x_2, x_3 and x_4, giving your answers to 4 decimal places.

(2)

(*d*) By considering the change of sign of $f'(x)$ in a suitable interval, prove that $\alpha = 0.1443$ correct to 4 decimal places.

(2)

2. $$f(x) = 2x^3 - x - 4.$$

(*a*) Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt{\left(\frac{2}{x} + \frac{1}{2}\right)}.$$

(3)

The equation $2x^3 - x - 4 = 0$ has a root between 1.35 and 1.4.

(*b*) Use the iteration formula

$$x_{n+1} = \sqrt{\left(\frac{2}{x_n} + \frac{1}{2}\right)},$$

with $x_0 = 1.35$, to find, to 2 decimal places, the value of x_1, x_2 and x_3.

(3)

The only real root of $f(x) = 0$ is α.

(*c*) By choosing a suitable interval, prove that $\alpha = 1.392$, to 3 decimal places.

(3)

3. **Figure 2**

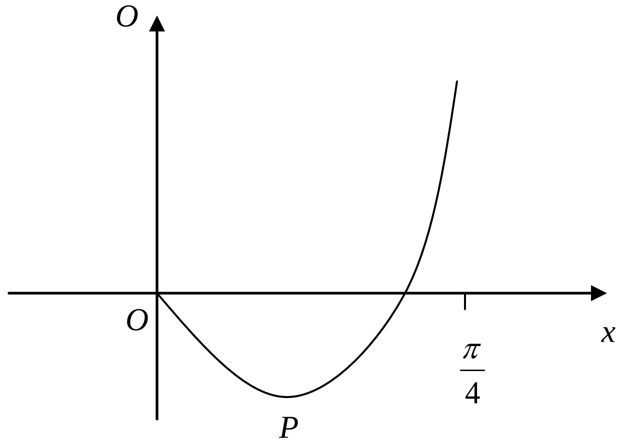


Figure 2 shows part of the curve with equation

$$y = (2x - 1)\tan 2x, \quad 0 \le x < \frac{\pi}{4}.$$

The curve has a minimum at the point *P*. The *x*-coordinate of *P* is *k*.

(*a*) Show that *k* satisfies the equation

$$4k + \sin 4k - 2 = 0.$$

(6)

The iterative formula

$$x_{n+1} = \tfrac{1}{4}(2 - \sin 4x_n), \quad x_0 = 0.3,$$

is used to find an approximate value for *k*.

(*b*) Calculate the values of x_1, x_2, x_3 and x_4, giving your answers to 4 decimals places.

(3)

(*c*) Show that $k = 0.277$, correct to 3 significant figures.

(2)

4. $$f(x) = \ln(x + 2) - x + 1, \qquad x > -2, x \in \mathbb{R}.$$

(*a*) Show that there is a root of $f(x) = 0$ in the interval $2 < x < 3$.

(2)

(*b*) Use the iterative formula

$$x_{n+1} = \ln(x_n + 2) + 1, \qquad x_0 = 2.5,$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 5 decimal places.

(3)

(*c*) Show that $x = 2.505$ is a root of $f(x) = 0$ correct to 3 decimal places.

(2)

5. $$f(x) = 3x^3 - 2x - 6.$$

(*a*) Show that $f(x) = 0$ has a root, α, between $x = 1.4$ and $x = 1.45$.

(2)

(*b*) Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt{\left(\frac{2}{x} + \frac{2}{3}\right)}, \quad x \neq 0.$$

(3)

(*c*) Starting with $x_0 = 1.43$, use the iteration

$$x_{n+1} = \sqrt{\left(\frac{2}{x_n} + \frac{2}{3}\right)}$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places.

(3)

(*d*) By choosing a suitable interval, show that $\alpha = 1.435$ is correct to 3 decimal places.

(3)

6. $$g(x) = e^{x-1} + x - 6$$

(*a*) Show that the equation $g(x) = 0$ can be written as

$$x = \ln(6 - x) + 1, \qquad x < 6.$$

(2)

The root of $g(x) = 0$ is α.

The iterative formula

$$x_{n+1} = \ln(6 - x_n) + 1, \qquad x_0 = 2.$$

is used to find an approximate value for α.

(*b*) Calculate the values of x_1, x_2 and x_3 to 4 decimal places.

(3)

(*c*) By choosing a suitable interval, show that $\alpha = 2.307$ correct to 3 decimal places.

(3)

7. $$f(x) = 2\sin(x^2) + x - 2, \qquad 0 \le x < 2\pi.$$

(*a*) Show that $f(x) = 0$ has a root α between $x = 0.75$ and $x = 0.85$.

(2)

The equation $f(x) = 0$ can be written as $x = [\arcsin(1 - 0.5x)]^{\frac{1}{2}}$.

(*b*) Use the iterative formula

$$x_{n+1} = [\arcsin(1 - 0.5x_n)]^{\frac{1}{2}}, \qquad x_0 = 0.8,$$

to find the values of x_1, x_2 and x_3, giving your answers to 5 decimal places.

(3)

(*c*) Show that $\alpha = 0.80157$ is correct to 5 decimal places.

(3)

Rsin(ϴ ± a)

Rcos(ϴ ± a)

Trigonometry	I know and can use the expressions for acosϴ + bsinϴ in the forms rcos(ϴ±a) and rsin(ϴ±a)	

1. (*a*) Using the identity $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$, prove that

$$\cos 2A \equiv 1 - 2\sin^2 A.$$

(2)

(*b*) Show that

$$2\sin 2\theta - 3\cos 2\theta - 3\sin\theta + 3 \equiv \sin\theta(4\cos\theta + 6\sin\theta - 3).$$

(4)

(*c*) Express $4\cos\theta + 6\sin\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$.

(4)

(*d*) Hence, for $0 \leq \theta < \pi$, solve

$$2\sin 2\theta = 3(\cos 2\theta + \sin\theta - 1),$$

giving your answers in radians to 3 significant figures, where appropriate.

(5)

2.

$$f(x) = 12\cos x - 4\sin x.$$

Given that $f(x) = R\cos(x + \alpha)$, where $R \geq 0$ and $0 \leq \alpha \leq 90°$,

(*a*) find the value of R and the value of α.

(4)

(*b*) Hence solve the equation

$$12\cos x - 4\sin x = 7$$

for $0 \leq x < 360°$, giving your answers to one decimal place.

(5)

(*c*) (i) Write down the minimum value of $12\cos x - 4\sin x$.

(1)

(ii) Find, to 2 decimal places, the smallest positive value of x for which this minimum value occurs.

(2)

3. **Figure 1**

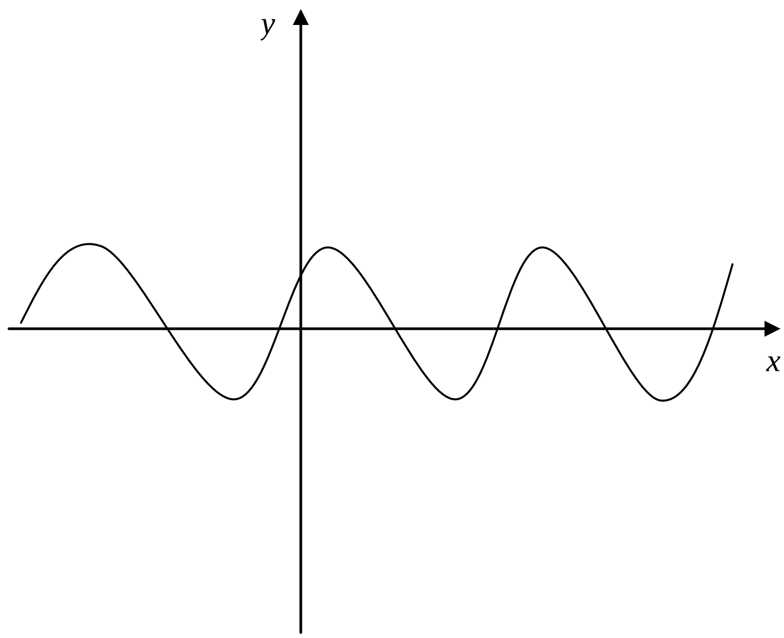


Figure 1 shows an oscilloscope screen.

The curve on the screen satisfies the equation $y = \sqrt{3}\cos x + \sin x$.

(*a*) Express the equation of the curve in the form $y = R\sin(x + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$.

(4)

(*b*) Find the values of x, $0 \le x < 2\pi$, for which $y = 1$.

(4)

4. (*a*) Use the double angle formulae and the identity

$$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$

to obtain an expression for $\cos 3x$ in terms of powers of $\cos x$ only.

(4)

(*b*) (i) Prove that

$$\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} \equiv 2 \sec x, \qquad x \neq (2n + 1)\frac{\pi}{2}.$$

(4)

(ii) Hence find, for $0 < x < 2\pi$, all the solutions of

$$\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} = 4.$$

(3)

5.

$$f(x) = 5 \cos x + 12 \sin x.$$

Given that $f(x) = R \cos (x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$,

(*a*) find the value of R and the value of α to 3 decimal places.

(4)

(*b*) Hence solve the equation

$$5 \cos x + 12 \sin x = 6$$

for $0 \leq x < 2\pi$.

(5)

(*c*) (i) Write down the maximum value of $5 \cos x + 12 \sin x$.

(1)

(ii) Find the smallest positive value of x for which this maximum value occurs.

(2)

6. (*a*) Express 6 cos θ + 8 sin θ in the form R cos ($\theta - \alpha$), where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 3 decimal places.

(4)

(*b*)
$$\mathrm{p}(\theta) = \frac{4}{12 + 6\cos\theta + 8\sin\theta}, \quad 0 \le \theta \le 2\pi.$$

Calculate

(i) the maximum value of p(θ),

(ii) the value of θ at which the maximum occurs.

(4)

7. (*a*) Express 2 cos 3x – 3 sin 3x in the form R cos (3x + α), where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give your answers to 3 significant figures.

(4)

$$\mathrm{f}(x) = \mathrm{e}^{2x} \cos 3x.$$

(*b*) Show that f ′(x) can be written in the form

$$\mathrm{f}'(x) = R\mathrm{e}^{2x} \cos (3x + \alpha),$$

where R and α are the constants found in part (*a*).

(5)

(*c*) Hence, or otherwise, find the smallest positive value of x for which the curve with equation y = f(x) has a turning point.

(3)

Coordinate Geometry and Transformations

Algebra and Functions	I know how to sketch the graph of $y = \|f(x)\|$.	
	I know how to sketch the graph of $y = f(\|x\|)$.	
	I can use combinations of transformations of $y=f(x)$ such as $af(x)$, $f(x)+a$, $f(x+a)$ and $y=f(ax)$.	
	I can identify x and y intercepts.	
	I can find normals and tangents.	

1. **Figure 1**

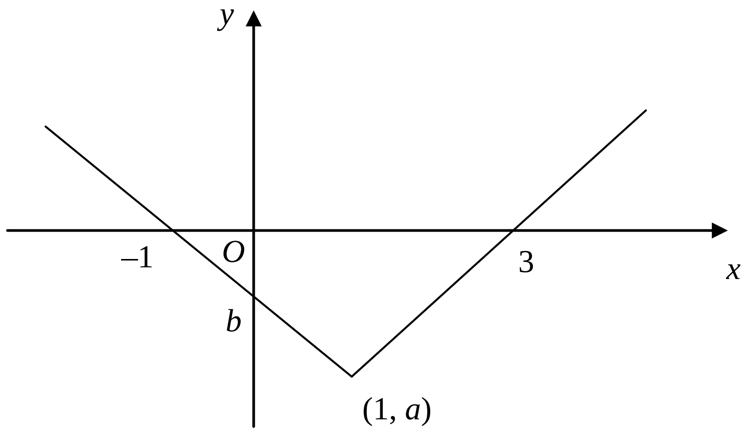


Figure 1 shows part of the graph of $y = f(x)$, $x \in \mathbb{R}$. The graph consists of two line segments that meet at the point $(1, a)$, $a < 0$. One line meets the x-axis at $(3, 0)$. The other line meets the x-axis at $(–1, 0)$ and the y-axis at $(0, b)$, $b < 0$.

In separate diagrams, sketch the graph with equation

(*a*) $y = f(x + 1)$,

(2)

(*b*) $y = f(|x|)$.

(3)

Indicate clearly on each sketch the coordinates of any points of intersection with the axes.

Given that $f(x) = |x – 1| – 2$, find

(*c*) the value of a and the value of b,

(2)

(*d*) the value of x for which $f(x) = 5x$.

(4)

2. **Figure 1**

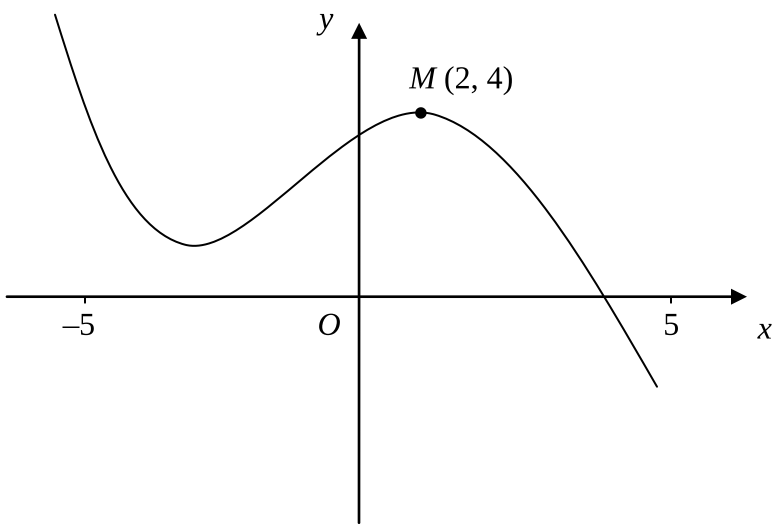


Figure 1 shows the graph of $y = f(x)$, $-5 \le x \le 5$.

The point M (2, 4) is the maximum turning point of the graph.

Sketch, on separate diagrams, the graphs of

(*a*) $y = f(x) + 3$,

(2)

(*b*) $y = |f(x)|$,

(2)

(*c*) $y = f(|x|)$.

(3)

Show on each graph the coordinates of any maximum turning points.

3.

Figure 1

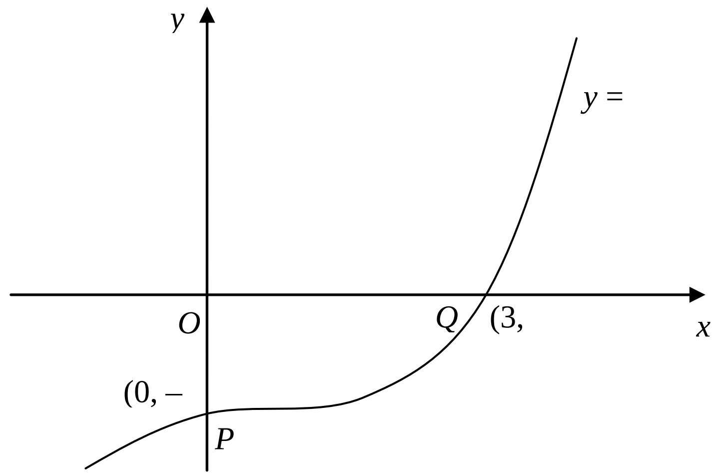


Figure 1 shows part of the curve with equation $y = \mathrm{f}(x)$, $x \in \mathbb{R}$, where f is an increasing function of x. The curve passes through the points $P(0, -2)$ and $Q(3, 0)$ as shown.

In separate diagrams, sketch the curve with equation

(*a*) $y = |\mathrm{f}(x)|$,

(3)

(*b*) $y = \mathrm{f}^{-1}(x)$,

(3)

(*c*) $y = \frac{1}{2}\mathrm{f}(3x)$.

(3)

Indicate clearly on each sketch the coordinates of the points at which the curve crosses or meets the axes.

4.

$$\mathrm{f}(x) = x^4 - 4x - 8.$$

(*a*) Show that there is a root of $\mathrm{f}(x) = 0$ in the interval $[-2, -1]$.

(3)

(*b*) Find the coordinates of the turning point on the graph of $y = \mathrm{f}(x)$.

(3)

(*c*) Given that $\mathrm{f}(x) = (x - 2)(x^3 + ax^2 + bx + c)$, find the values of the constants a, b and c.

(3)

(*d*) Sketch the graph of $y = \mathrm{f}(x)$.

(3)

(*e*) Hence sketch the graph of $y = |\mathrm{f}(x)|$.

(1)

5. For the constant k, where $k > 1$, the functions f and g are defined by

$$\text{f}: x \mapsto \ln(x + k), \quad x > -k,$$

$$\text{g}: x \mapsto |2x - k|, \quad x \in \mathbb{R}.$$

(a) On separate axes, sketch the graph of f and the graph of g.

On each sketch state, in terms of k, the coordinates of points where the graph meets the coordinate axes.

(5)

(b) Write down the range of f.

(1)

(c) Find $\text{fg}\left(\frac{k}{4}\right)$ in terms of k, giving your answer in its simplest form.

(2)

The curve C has equation $y = \text{f}(x)$. The tangent to C at the point with x-coordinate 3 is parallel to the line with equation $9y = 2x + 1$.

(d) Find the value of k.

(4)

6. A curve C has equation

$$y = 3\sin 2x + 4\cos 2x, \qquad -\pi \le x \le \pi .$$

The point $A(0, 4)$ lies on C.

(a) Find an equation of the normal to the curve C at A.

(5)

(b) Express y in the form $R\sin(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 3 significant figures.

(4)

(c) Find the coordinates of the points of intersection of the curve C with the x-axis.
Give your answers to 2 decimal places.

(4)

7.

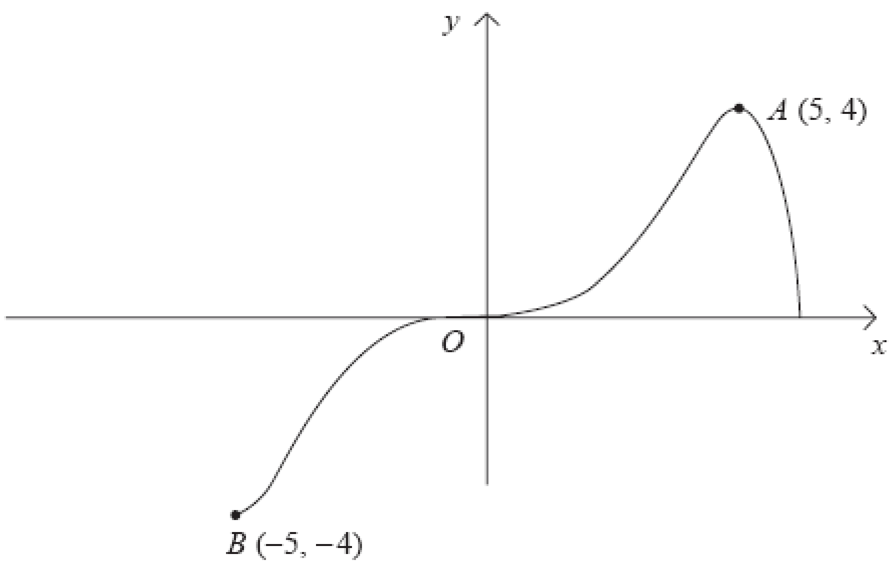


Figure 1

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$.

The curve passes through the origin O and the points $A(5, 4)$ and $B(–5, –4)$.

In separate diagrams, sketch the graph with equation

(*a*) $y = |\mathrm{f}(x)|$,

(3)

(*b*) $y = \mathrm{f}(|x|)$,

(3)

(*c*) $y = 2\mathrm{f}(x + 1)$.

(4)

On each sketch, show the coordinates of the points corresponding to A and B.

8. The point P lies on the curve with equation

$$y = 4e^{2x + 1}.$$

The y-coordinate of P is 8.

(*a*) Find, in terms of ln 2, the x-coordinate of P.

(2)

(*b*) Find the equation of the tangent to the curve at the point P in the form $y = ax + b$, where a and b are exact constants to be found.

(4)

9.

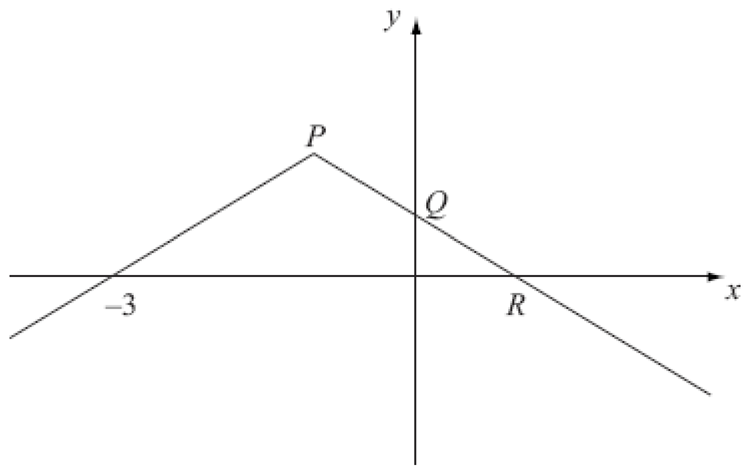


Figure 1

Figure 1 shows the graph of $y = \mathrm{f}(x)$, $x \in \mathbb{R}$,

The graph consists of two line segments that meet at the point P.

The graph cuts the y-axis at the point Q and the x-axis at the points (–3, 0) and R.

Sketch, on separate diagrams, the graphs of

(*a*) $y = |\mathrm{f}(x)|$,

(2)

(*b*) $y = \mathrm{f}(-x)$.

(2)

Given that $\mathrm{f}(x) = 2 - |x + 1|$,

(*c*) find the coordinates of the points P, Q and R,

(3)

(*d*) solve $\mathrm{f}(x) = \frac{1}{2}x$.

(5)

10.

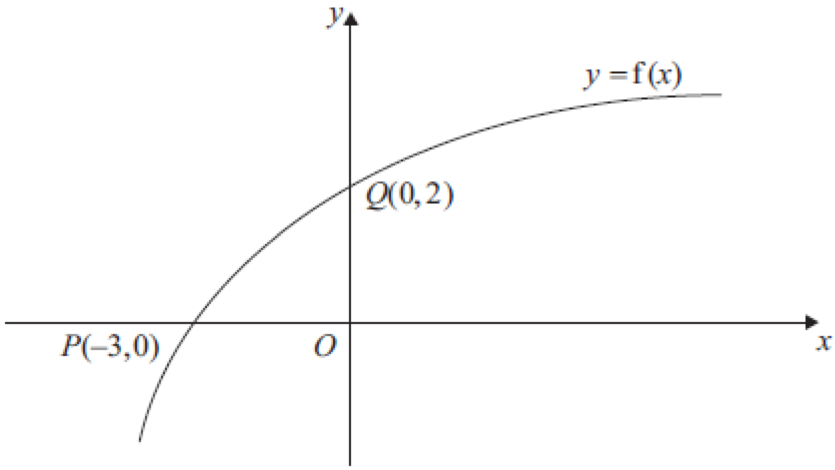


Figure 1

Figure 1 shows part of the curve with equation $y = \mathrm{f}(x)$, $x \in \mathbb{R}$.

The curve passes through the points $Q(0, 2)$ and $P(-3, 0)$ as shown.

(*a*) Find the value of ff (-3).

(2)

On separate diagrams, sketch the curve with equation

(*b*) $y = \mathrm{f}^{-1}(x)$,

(2)

(*c*) $y = \mathrm{f}(|x|) - 2$,

(2)

(*d*) $y = 2\mathrm{f}\left(\frac{1}{2}x\right)$.

(3)

Indicate clearly on each sketch the coordinates of the points at which the curve crosses or meets the axes.

11.

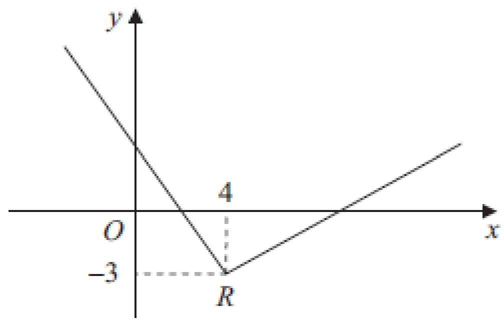


Figure 1

Figure 1 shows part of the graph of $y = \mathrm{f}(x)$, $x \in \mathbb{R}$.

The graph consists of two line segments that meet at the point R (4, – 3), as shown in Figure 1.

Sketch, on separate diagrams, the graphs of

(*a*) $y = 2\mathrm{f}(x + 4)$,

(3)

(*b*) $y = |\mathrm{f}(-x)|$.

(3)

On each diagram, show the coordinates of the point corresponding to R.

Exponential Equations

Exponentials and Logarithms	I know the function e^x and its graph.	
	I understand how the graph of e^x may be transformed, eg: $e^{ax + b} + c$.	
	I understand the function ln(x) and its graph.	
	I understand the relationship between ln(x) and e^x.	
	I can solve equations of the form $e^{ax + b} = p$.	
	I can solve equations of the form $\ln(ax + b) = q$.	

1. A particular species of orchid is being studied. The population p at time t years after the study started is assumed to be

$$p = \frac{2800ae^{0.2t}}{1 + ae^{0.2t}}, \text{ where } a \text{ is a constant.}$$

Given that there were 300 orchids when the study started,

(*a*) show that $a = 0.12$,

(3)

(*b*) use the equation with $a = 0.12$ to predict the number of years before the population of orchids reaches 1850.

(4)

(*c*) Show that $p = \dfrac{336}{0.12 + e^{-0.2t}}$.

(1)

(*d*) Hence show that the population cannot exceed 2800.

(2)

2. A heated metal ball is dropped into a liquid. As the ball cools, its temperature, T °C, t minutes after it enters the liquid, is given by

$$T = 400e^{-0.05t} + 25, \quad t \geq 0.$$

(*a*) Find the temperature of the ball as it enters the liquid.

(1)

(*b*) Find the value of t for which $T = 300$, giving your answer to 3 significant figures.

(4)

(*c*) Find the rate at which the temperature of the ball is decreasing at the instant when $t = 50$. Give your answer in °C per minute to 3 significant figures.

(3)

(*d*) From the equation for temperature T in terms of t, given above, explain why the temperature of the ball can never fall to 20 °C.

(1)

3. The radioactive decay of a substance is given by

$$R = 1000e^{-ct}, \quad t \geq 0.$$

where R is the number of atoms at time t years and c is a positive constant.

(*a*) Find the number of atoms when the substance started to decay.
(1)

It takes 5730 years for half of the substance to decay.

(*b*) Find the value of c to 3 significant figures.
(4)

(*c*) Calculate the number of atoms that will be left when t = 22 920 .
(2)

(*d*) Sketch the graph of R against t .
(2)

4. The value of Bob's car can be calculated from the formula

$$V = 17000e^{-0.25t} + 2000e^{-0.5t} + 500.$$

where V is the value of the car in pounds (£) and t is the age in years.

(*a*) Find the value of the car when $t = 0$.
(1)

(*b*) Calculate the exact value of t when $V = 9500$.
(4)

(*c*) Find the rate at which the value of the car is decreasing at the instant when $t = 8$.
Give your answer in pounds per year to the nearest pound.
(4)

5. The mass, m grams, of a leaf t days after it has been picked from a tree is given by

$$m = pe^{-kt},$$

where k and p are positive constants.

When the leaf is picked from the tree, its mass is 7.5 grams and 4 days later its mass is 2.5 grams.

(*a*) Write down the value of p.
(1)

(*b*) Show that $k = \frac{1}{4}\ln 3$.
(4)

(*c*) Find the value of t when $\frac{dm}{dt} = -0.6 \ln 3$.
(6)