Further Pure 1
OCR Past Papers

June 2005

1 Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n,

$$\sum_{r=1}^{n}(6r^2 + 2r + 1) = n(2n^2 + 4n + 3).$$ **[6]**

2 The matrices $\mathbf{A}$ and $\mathbf{I}$ are given by $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ respectively.

(i) Find $\mathbf{A}^2$ and verify that $\mathbf{A}^2 = 4\mathbf{A} - \mathbf{I}$. [4]

(ii) Hence, or otherwise, show that $\mathbf{A}^{-1} = 4\mathbf{I} - \mathbf{A}$. [2]

3 The complex numbers $2 + 3i$ and $4 - i$ are denoted by z and w respectively. Express each of the following in the form $x + iy$, showing clearly how you obtain your answers.

(i) $z + 5w$, [2]

(ii) z^*w, where z^* is the complex conjugate of z, [3]

(iii) $\frac{1}{w}$. [2]

4 Use an algebraic method to find the square roots of the complex number $21 - 20i$. [6]

5 **(i)** Show that

$$\frac{r+1}{r+2} - \frac{r}{r+1} = \frac{1}{(r+1)(r+2)}.$$ [2]

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots + \frac{1}{(n+1)(n+2)}.$$ **[4]**

(iii) Hence write down the value of $\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+2)}$. [1]

6 The loci C_1 and C_2 are given by

$$|z - 2i| = 2 \qquad \text{and} \qquad |z + 1| = |z + i|$$

respectively.

(i) Sketch, on a single Argand diagram, the loci C_1 and C_2. [5]

(ii) Hence write down the complex numbers represented by the points of intersection of C_1 and C_2. [2]

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7 The matrix **B** is given by $\mathbf{B} = \begin{pmatrix} a & 1 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$.

(i) Given that **B** is singular, show that $a = -\frac{2}{3}$. [3]

(ii) Given instead that **B** is non-singular, find the inverse matrix $\mathbf{B}^{-1}$. [4]

(iii) Hence, or otherwise, solve the equations

$$\begin{aligned} -x + y + 3z &= 1, \\ 2x + y - z &= 4, \\ y + 2z &= -1. \end{aligned}$$ [3]

8 **(a)** The quadratic equation $x^2 - 2x + 4 = 0$ has roots α and β.

(i) Write down the values of $\alpha + \beta$ and $\alpha\beta$. [2]

(ii) Show that $\alpha^2 + \beta^2 = -4$. [2]

(iii) Hence find a quadratic equation which has roots α^2 and β^2. [3]

(b) The cubic equation $x^3 - 12x^2 + ax - 48 = 0$ has roots p, $2p$ and $3p$.

(i) Find the value of p. [2]

(ii) Hence find the value of a. [2]

9 **(i)** Write down the matrix **C** which represents a stretch, scale factor 2, in the x-direction. [2]

(ii) The matrix **D** is given by $\mathbf{D} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. Describe fully the geometrical transformation represented by **D**. [2]

(iii) The matrix **M** represents the combined effect of the transformation represented by **C** followed by the transformation represented by **D**. Show that

$$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}.$$ [2]

(iv) Prove by induction that $\mathbf{M}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}$, for all positive integers n. [6]

Jan 2006

1 **(i)** Express $(1+8\text{i})(2-\text{i})$ in the form $x+\text{i}y$, showing clearly how you obtain your answer. [2]

(ii) Hence express $\frac{1+8\text{i}}{2+\text{i}}$ in the form $x+\text{i}y$. [3]

2 Prove by induction that, for $n \geq 1$, $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [5]

3 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.

(i) Find the value of the determinant of **M**. [3]

(ii) State, giving a brief reason, whether **M** is singular or non-singular. [1]

4 Use the substitution $x = u + 2$ to find the exact value of the real root of the equation

$$x^3 - 6x^2 + 12x - 13 = 0.$$ [5]

5 Use the standard results for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ to show that, for all positive integers n,

$$\sum_{r=1}^{n}(8r^3 - 6r^2 + 2r) = 2n^3(n+1).$$ **[6]**

6 The matrix **C** is given by $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$.

(i) Find $\mathbf{C}^{-1}$. [2]

(ii) Given that $\mathbf{C} = \mathbf{AB}$, where $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, find $\mathbf{B}^{-1}$. [5]

7 **(a)** The complex number $3+2\text{i}$ is denoted by w and the complex conjugate of w is denoted by w^*. Find

(i) the modulus of w, [1]

(ii) the argument of w^*, giving your answer in radians, correct to 2 decimal places. [3]

(b) Find the complex number u given that $u + 2u^* = 3 + 2\text{i}$. [4]

(c) Sketch, on an Argand diagram, the locus given by $|z+1| = |z|$. [2]

Jan 2006

8 The matrix $\mathbf{T}$ is given by $\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$.

(i) Draw a diagram showing the unit square and its image under the transformation represented by $\mathbf{T}$. [3]

(ii) The transformation represented by matrix $\mathbf{T}$ is equivalent to a transformation A, followed by a transformation B. Give geometrical descriptions of possible transformations A and B, and state the matrices that represent them. [6]

9 **(i)** Show that $\frac{1}{r} - \frac{1}{r+2} = \frac{2}{r(r+2)}$. [2]

(ii) Hence find an expression, in terms of n, for

$$\frac{2}{1\times 3} + \frac{2}{2\times 4} + \dots + \frac{2}{n(n+2)}.$$ [5]

(iii) Hence find the value of

(a) $\sum_{r=1}^{\infty} \frac{2}{r(r+2)}$, [1]

(b) $\sum_{r=n+1}^{\infty} \frac{2}{r(r+2)}$. [2]

10 The roots of the equation

$$x^3 - 9x^2 + 27x - 29 = 0$$

are denoted by α, β and γ, where α is real and β and γ are complex.

(i) Write down the value of $\alpha + \beta + \gamma$. [1]

(ii) It is given that $\beta = p + iq$, where $q > 0$. Find the value of p, in terms of α. [4]

(iii) Write down the value of $\alpha\beta\gamma$. [1]

(iv) Find the value of q, in terms of α only. [5]

June 2006

1 The matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

(i) Find $\mathbf{A} + 3\mathbf{B}$. [2]

(ii) Show that $\mathbf{A} - \mathbf{B} = k\mathbf{I}$, where **I** is the identity matrix and k is a constant whose value should be stated. [2]

2 The transformation S is a shear parallel to the x-axis in which the image of the point (1, 1) is the point (0, 1).

(i) Draw a diagram showing the image of the unit square under S. [2]

(ii) Write down the matrix that represents S. [2]

3 One root of the quadratic equation $x^2 + px + q = 0$, where p and q are real, is the complex number $2 - 3\text{i}$.

(i) Write down the other root. [1]

(ii) Find the values of p and q. [4]

4 Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} (r^3 + r^2) = \frac{1}{12}n(n+1)(n+2)(3n+1).$$ [5]

5 The complex numbers $3 - 2\text{i}$ and $2 + \text{i}$ are denoted by z and w respectively. Find, giving your answers in the form $x + \text{i}y$ and showing clearly how you obtain these answers,

(i) $2z - 3w$, [2]

(ii) $(\text{i}z)^2$, [3]

(iii) $\dfrac{z}{w}$. [3]

6 In an Argand diagram the loci C_1 and C_2 are given by

$$|z| = 2 \qquad \text{and} \qquad \arg z = \tfrac{1}{3}\pi$$

respectively.

(i) Sketch, on a single Argand diagram, the loci C_1 and C_2. [5]

(ii) Hence find, in the form $x + \text{i}y$, the complex number representing the point of intersection of C_1 and C_2. [2]

June 2006

7 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

(i) Find $\mathbf{A}^2$ and $\mathbf{A}^3$. [3]

(ii) Hence suggest a suitable form for the matrix $\mathbf{A}^n$. [1]

(iii) Use induction to prove that your answer to part **(ii)** is correct. [4]

8 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} a & 4 & 2 \\ 1 & a & 0 \\ 1 & 2 & 1 \end{pmatrix}$.

(i) Find, in terms of a, the determinant of **M**. [3]

(ii) Hence find the values of a for which **M** is singular. [3]

(iii) State, giving a brief reason in each case, whether the simultaneous equations

$$\begin{aligned} ax + 4y + 2z &= 3a, \\ x + ay &= 1, \\ x + 2y + z &= 3, \end{aligned}$$

have any solutions when

(a) $a = 3$,

(b) $a = 2$.

[4]

9 **(i)** Use the method of differences to show that

$$\sum_{r=1}^{n} \{(r+1)^3 - r^3\} = (n+1)^3 - 1.$$ [2]

(ii) Show that $(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1$. [2]

(iii) Use the results in parts **(i)** and **(ii)** and the standard result for $\sum_{r=1}^{n} r$ to show that

$$3\sum_{r=1}^{n} r^2 = \frac{1}{2}n(n+1)(2n+1).$$ [6]

10 The cubic equation $x^3 - 2x^2 + 3x + 4 = 0$ has roots α, β and γ.

(i) Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. [3]

The cubic equation $x^3 + px^2 + 10x + q = 0$, where p and q are constants, has roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$.

(ii) Find the value of p. [3]

(iii) Find the value of q. [5]

1 The matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & -1 \\ -3 & -2 \end{pmatrix}$.

(i) Given that $2\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$, write down the value of a. [1]

(ii) Given instead that $\mathbf{AB} = \begin{pmatrix} 7 & -4 \\ 9 & -7 \end{pmatrix}$, find the value of a. [2]

2 Use an algebraic method to find the square roots of the complex number $15 + 8\mathrm{i}$. [6]

3 Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ to find

$$\sum_{r=1}^{n} r(r-1)(r+1),$$

expressing your answer in a fully factorised form. [6]

4 **(i)** Sketch, on an Argand diagram, the locus given by $|z - 1 + \mathrm{i}| = \sqrt{2}$. [3]

(ii) Shade on your diagram the region given by $1 \leq |z - 1 + \mathrm{i}| \leq \sqrt{2}$. [3]

5 **(i)** Verify that $z^3 - 8 = (z-2)(z^2 + 2z + 4)$. [1]

(ii) Solve the quadratic equation $z^2 + 2z + 4 = 0$, giving your answers exactly in the form $x + \mathrm{i}y$. Show clearly how you obtain your answers. [3]

(iii) Show on an Argand diagram the roots of the cubic equation $z^3 - 8 = 0$. [3]

6 The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_n = n^2 + 3n$, for all positive integers n.

(i) Show that $u_{n+1} - u_n = 2n + 4$. [3]

(ii) Hence prove by induction that each term of the sequence is divisible by 2. [5]

7 The quadratic equation $x^2 + 5x + 10 = 0$ has roots α and β.

(i) Write down the values of $\alpha + \beta$ and $\alpha\beta$. [2]

(ii) Show that $\alpha^2 + \beta^2 = 5$. [2]

(iii) Hence find a quadratic equation which has roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$. [4]

Jan 2007

8 **(i)** Show that $(r+2)! - (r+1)! = (r+1)^2 \times r!$. [3]

(ii) Hence find an expression, in terms of n, for

$$2^2 \times 1! + 3^2 \times 2! + 4^2 \times 3! + \ldots + (n+1)^2 \times n!.$$ [4]

(iii) State, giving a brief reason, whether the series

$$2^2 \times 1! + 3^2 \times 2! + 4^2 \times 3! + \ldots$$

converges. [1]

9 The matrix **C** is given by $\mathbf{C} = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$.

(i) Draw a diagram showing the unit square and its image under the transformation represented by **C**. [2]

The transformation represented by **C** is equivalent to a rotation, R, followed by another transformation, S.

(ii) Describe fully the rotation R and write down the matrix that represents R. [3]

(iii) Describe fully the transformation S and write down the matrix that represents S. [4]

10 The matrix **D** is given by $\mathbf{D} = \begin{pmatrix} a & 2 & 0 \\ 3 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix}$, where $a \neq 2$.

(i) Find $\mathbf{D}^{-1}$. [7]

(ii) Hence, or otherwise, solve the equations

$$\begin{aligned} ax + 2y &= 3, \\ 3x + y + 2z &= 4, \\ -y + z &= 1. \end{aligned}$$ [4]

1 The complex number $a + \mathrm{i}b$ is denoted by z. Given that $|z| = 4$ and $\arg z = \frac{1}{3}\pi$, find a and b. [4]

2 Prove by induction that, for $n \geq 1$, $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$. [5]

3 Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n,

$$\sum_{r=1}^{n}(3r^2 - 3r + 1) = n^3.$$ [6]

4 The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$.

(i) Find $\mathbf{A}^{-1}$. [2]

The matrix $\mathbf{B}^{-1}$ is given by $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$.

(ii) Find $(\mathbf{AB})^{-1}$. [4]

5 **(i)** Show that

$$\frac{1}{r} - \frac{1}{r+1} = \frac{1}{r(r+1)}.$$ [1]

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{n(n+1)}.$$ [3]

(iii) Hence find the value of $\sum_{r=n+1}^{\infty} \frac{1}{r(r+1)}$. [3]

6 The cubic equation $3x^3 - 9x^2 + 6x + 2 = 0$ has roots α, β and γ.

(i) (a) Write down the values of $\alpha + \beta + \gamma$ and $\alpha\beta + \beta\gamma + \gamma\alpha$. [2]

(b) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(ii) (a) Use the substitution $x = \frac{1}{u}$ to find a cubic equation in u with integer coefficients. [2]

(b) Use your answer to part **(ii) (a)** to find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. [2]

June 2007

7 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} a & 4 & 0 \\ 0 & a & 4 \\ 2 & 3 & 1 \end{pmatrix}$.

(i) Find, in terms of a, the determinant of **M**. [3]

(ii) In the case when $a = 2$, state whether **M** is singular or non-singular, justifying your answer. [2]

(iii) In the case when $a = 4$, determine whether the simultaneous equations

$$\begin{aligned} ax + 4y \quad &= 6, \\ ay + 4z &= 8, \\ 2x + 3y + z &= 1, \end{aligned}$$

have any solutions. [3]

8 The loci C_1 and C_2 are given by $|z - 3| = 3$ and $\arg(z - 1) = \frac{1}{4}\pi$ respectively.

(i) Sketch, on a single Argand diagram, the loci C_1 and C_2. [6]

(ii) Indicate, by shading, the region of the Argand diagram for which

$$|z - 3| \leq 3 \text{ and } 0 \leq \arg(z - 1) \leq \tfrac{1}{4}\pi.$$ [2]

9 **(i)** Write down the matrix, **A**, that represents an enlargement, centre (0, 0), with scale factor $\sqrt{2}$. [1]

(ii) The matrix **B** is given by $\mathbf{B} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$. Describe fully the geometrical transformation represented by **B**. [3]

(iii) Given that **C** = **AB**, show that $\mathbf{C} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. [1]

(iv) Draw a diagram showing the unit square and its image under the transformation represented by **C**. [2]

(v) Write down the determinant of **C** and explain briefly how this value relates to the transformation represented by **C**. [2]

10 **(i)** Use an algebraic method to find the square roots of the complex number $16 + 30\mathrm{i}$. [6]

(ii) Use your answers to part **(i)** to solve the equation $z^2 - 2z - (15 + 30\mathrm{i}) = 0$, giving your answers in the form $x + \mathrm{i}y$. [5]

Jan 2008

1 The transformation S is a shear with the y-axis invariant (i.e. a shear parallel to the y-axis). It is given that the image of the point $(1, 1)$ is the point $(1, 0)$.

(i) Draw a diagram showing the image of the unit square under the transformation S. [2]

(ii) Write down the matrix that represents S. [2]

2 Given that $\sum_{r=1}^{n}(ar^2 + b) \equiv n(2n^2 + 3n - 2)$, find the values of the constants a and b. [5]

3 The cubic equation $2x^3 - 3x^2 + 24x + 7 = 0$ has roots α, β and γ.

(i) Use the substitution $x = \frac{1}{u}$ to find a cubic equation in u with integer coefficients. [2]

(ii) Hence, or otherwise, find the value of $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$. [2]

4 The complex number $3 - 4\text{i}$ is denoted by z. Giving your answers in the form $x + \text{i}y$, and showing clearly how you obtain them, find

(i) $2z + 5z^*$, [2]

(ii) $(z - \text{i})^2$, [3]

(iii) $\frac{3}{z}$. [3]

5 The matrices **A**, **B** and **C** are given by $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{C} = (2 \quad 4 \quad -1)$. Find

(i) $\mathbf{A} - 4\mathbf{B}$, [2]

(ii) $\mathbf{BC}$, [4]

(iii) $\mathbf{CA}$. [2]

6 The loci C_1 and C_2 are given by

$$|z| = |z - 4\text{i}| \qquad \text{and} \qquad \arg z = \tfrac{1}{6}\pi$$

respectively.

(i) Sketch, on a single Argand diagram, the loci C_1 and C_2. [5]

(ii) Hence find, in the form $x + \text{i}y$, the complex number represented by the point of intersection of C_1 and C_2. [3]

Jan 2008

7 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} a & 3 \\ -2 & 1 \end{pmatrix}$.

(i) Given that **A** is singular, find a. [2]

(ii) Given instead that **A** is non-singular, find $\mathbf{A}^{-1}$ and hence solve the simultaneous equations

$$ax + 3y = 1,$$
$$-2x + y = -1.$$ [5]

8 The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 1$ and $u_{n+1} = u_n + 2n + 1$.

(i) Show that $u_4 = 16$. [2]

(ii) Hence suggest an expression for u_n. [1]

(iii) Use induction to prove that your answer to part **(ii)** is correct. [4]

9 **(i)** Show that $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$. [2]

(ii) The quadratic equation $x^2 - 5x + 7 = 0$ has roots α and β. Find a quadratic equation with roots α^3 and β^3. [6]

10 **(i)** Show that $\dfrac{2}{r} - \dfrac{1}{r+1} - \dfrac{1}{r+2} = \dfrac{3r+4}{r(r+1)(r+2)}$. [2]

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=1}^{n} \frac{3r+4}{r(r+1)(r+2)}.$$ [6]

(iii) Hence write down the value of $\displaystyle\sum_{r=1}^{\infty} \frac{3r+4}{r(r+1)(r+2)}$. [1]

(iv) Given that $\displaystyle\sum_{r=N+1}^{\infty} \frac{3r+4}{r(r+1)(r+2)} = \frac{7}{10}$, find the value of N. [4]

Jan 2008

1 The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix}$ and $\mathbf{I}$ is the 2×2 identity matrix. Find

(i) $\mathbf{A} - 3\mathbf{I}$, [2]

(ii) $\mathbf{A}^{-1}$. [2]

2 The complex number $3 + 4\mathrm{i}$ is denoted by a.

(i) Find $|a|$ and $\arg a$. [2]

(ii) Sketch on a single Argand diagram the loci given by

(a) $|z - a| = |a|$, [2]

(b) $\arg(z - 3) = \arg a$. [3]

3 **(i)** Show that $\dfrac{1}{r!} - \dfrac{1}{(r+1)!} = \dfrac{r}{(r+1)!}$. [2]

(ii) Hence find an expression, in terms of n, for

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}.$$ [4]

4 The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for $n \geq 1$,

$$\mathbf{A}^n = \begin{pmatrix} 3^n & \frac{1}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}.$$ [6]

5 Find $\displaystyle\sum_{r=1}^{n} r^2(r-1)$, expressing your answer in a fully factorised form. [6]

6 The cubic equation $x^3 + ax^2 + bx + c = 0$, where a, b and c are real, has roots $(3 + \mathrm{i})$ and 2.

(i) Write down the other root of the equation. [1]

(ii) Find the values of a, b and c. [6]

Jan 2008

7 Describe fully the geometrical transformation represented by each of the following matrices:

(i) $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$, [1]

(ii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, [2]

(iii) $\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$, [2]

(iv) $\begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix}$. [2]

8 The quadratic equation $x^2 + kx + 2k = 0$, where k is a non-zero constant, has roots α and β. Find a quadratic equation with roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$. [7]

9 **(i)** Use an algebraic method to find the square roots of the complex number $5 + 12i$. [5]

(ii) Find $(3 - 2i)^2$. [2]

(iii) Hence solve the quartic equation $x^4 - 10x^2 + 169 = 0$. [4]

10 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} a & 8 & 10 \\ 2 & 1 & 2 \\ 4 & 3 & 6 \end{pmatrix}$. The matrix **B** is such that $\mathbf{AB} = \begin{pmatrix} a & 6 & 1 \\ 1 & 1 & 0 \\ 1 & 3 & 0 \end{pmatrix}$.

(i) Show that **AB** is non-singular. [2]

(ii) Find $(\mathbf{AB})^{-1}$. [4]

(iii) Find $\mathbf{B}^{-1}$. [5]

Jan 2009

1 Express $\frac{2+3\mathrm{i}}{5-\mathrm{i}}$ in the form $x + \mathrm{i}y$, showing clearly how you obtain your answer. **[4]**

2 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ a & 5 \end{pmatrix}$. Find

(i) $\mathbf{A}^{-1}$, **[2]**

(ii) $2\mathbf{A} - \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$. **[2]**

3 Find $\sum_{r=1}^{n}(4r^3 + 6r^2 + 2r)$, expressing your answer in a fully factorised form. **[6]**

4 Given that **A** and **B** are 2 × 2 non-singular matrices and **I** is the 2 × 2 identity matrix, simplify

$$\mathbf{B}(\mathbf{AB})^{-1}\mathbf{A} - \mathbf{I}.$$ **[4]**

5 By using the determinant of an appropriate matrix, or otherwise, find the value of k for which the simultaneous equations

$$\begin{aligned} 2x - y + z &= 7, \\ 3y + z &= 4, \\ x + ky + kz &= 5, \end{aligned}$$

do not have a unique solution for x, y and z. **[5]**

6 **(i)** The transformation P is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Give a geometrical description of transformation P. **[2]**

(ii) The transformation Q is represented by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Give a geometrical description of transformation Q. **[2]**

(iii) The transformation R is equivalent to transformation P followed by transformation Q. Find the matrix that represents R. **[2]**

(iv) Give a geometrical description of the **single** transformation that is represented by your answer to part **(iii)**. **[3]**

7 It is given that $u_n = 13^n + 6^{n-1}$, where n is a positive integer.

(i) Show that $u_n + u_{n+1} = 14 \times 13^n + 7 \times 6^{n-1}$. **[3]**

(ii) Prove by induction that u_n is a multiple of 7. **[4]**

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8 **(i)** Show that $(\alpha - \beta)^2 \equiv (\alpha + \beta)^2 - 4\alpha\beta$. **[2]**

The quadratic equation $x^2 - 6kx + k^2 = 0$, where k is a positive constant, has roots α and β, with $\alpha > \beta$.

(ii) Show that $\alpha - \beta = 4\sqrt{2}k$. **[4]**

(iii) Hence find a quadratic equation with roots $\alpha + 1$ and $\beta - 1$. **[4]**

9 **(i)** Show that $\dfrac{1}{2r-3} - \dfrac{1}{2r+1} = \dfrac{4}{4r^2 - 4r - 3}$. **[2]**

(ii) Hence find an expression, in terms of n, for

$$\sum_{r=2}^{n} \frac{4}{4r^2 - 4r - 3}.$$ **[6]**

(iii) Show that $\displaystyle\sum_{r=2}^{\infty} \frac{4}{4r^2 - 4r - 3} = \frac{4}{3}$. **[1]**

10 **(i)** Use an algebraic method to find the square roots of the complex number $2 + i\sqrt{5}$. Give your answers in the form $x + iy$, where x and y are exact real numbers. **[6]**

(ii) Hence find, in the form $x + iy$ where x and y are exact real numbers, the roots of the equation

$$z^4 - 4z^2 + 9 = 0.$$ **[4]**

(iii) Show, on an Argand diagram, the roots of the equation in part **(ii)**. **[1]**

(iv) Given that α is the root of the equation in part **(ii)** such that $0 < \arg \alpha < \frac{1}{2}\pi$, sketch on the same Argand diagram the locus given by $|z - \alpha| = |z|$. **[3]**

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1 Evaluate $\sum_{r=101}^{250} r^3$. **[3]**

2 The matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$ and **I** is the 2×2 identity matrix. Find the values of the constants a and b for which $a\mathbf{A} + b\mathbf{B} = \mathbf{I}$. **[4]**

3 The complex numbers z and w are given by $z = 5 - 2i$ and $w = 3 + 7i$. Giving your answers in the form $x + iy$ and showing clearly how you obtain them, find

(i) $4z - 3w$, **[2]**

(ii) z^*w. **[2]**

4 The roots of the quadratic equation $x^2 + x - 8 = 0$ are p and q. Find the value of $p + q + \frac{1}{p} + \frac{1}{q}$. **[4]**

5 The cubic equation $x^3 + 5x^2 + 7 = 0$ has roots α, β and γ.

(i) Use the substitution $x = \sqrt{u}$ to find a cubic equation in u with integer coefficients. **[3]**

(ii) Hence find the value of $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$. **[2]**

6 The complex number $3 - 3i$ is denoted by a.

(i) Find $|a|$ and $\arg a$. **[2]**

(ii) Sketch on a single Argand diagram the loci given by

(a) $|z - a| = 3\sqrt{2}$, **[3]**

(b) $\arg(z - a) = \frac{1}{4}\pi$. **[3]**

(iii) Indicate, by shading, the region of the Argand diagram for which

$$|z - a| \geq 3\sqrt{2} \quad \text{and} \quad 0 \leq \arg(z - a) \leq \tfrac{1}{4}\pi.$$ **[3]**

7 **(i)** Use the method of differences to show that

$$\sum_{r=1}^{n} \{(r + 1)^4 - r^4\} = (n + 1)^4 - 1.$$ **[2]**

(ii) Show that $(r + 1)^4 - r^4 \equiv 4r^3 + 6r^2 + 4r + 1$. **[2]**

(iii) Hence show that

$$4\sum_{r=1}^{n} r^3 = n^2(n + 1)^2.$$ **[6]**

8 The matrix **C** is given by $\mathbf{C} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$.

(i) Draw a diagram showing the image of the unit square under the transformation represented by **C**. **[3]**

The transformation represented by **C** is equivalent to a transformation S followed by another transformation T.

(ii) Given that S is a shear with the y-axis invariant in which the image of the point $(1, 1)$ is $(1, 2)$, write down the matrix that represents S. **[2]**

(iii) Find the matrix that represents transformation T and describe fully the transformation T. **[6]**

9 The matrix **A** is given by $\mathbf{A} = \begin{matrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 2 \end{matrix}$.

(i) Find, in terms of a, the determinant of **A**. **[3]**

(ii) Hence find the values of a for which **A** is singular. **[3]**

(iii) State, giving a brief reason in each case, whether the simultaneous equations

$$ax + y + z = 2a,$$
$$x + ay + z = -1,$$
$$x + y + 2z = -1,$$

have any solutions when

(a) $a = 0$,

(b) $a = 1$.

[4]

10 The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 3$ and $u_{n+1} = 3u_n - 2$.

(i) Find u_2 and u_3 and verify that $\frac{1}{2}(u_4 - 1) = 27$. **[3]**

(ii) Hence suggest an expression for u_n. **[2]**

(iii) Use induction to prove that your answer to part **(ii)** is correct. **[5]**