

Pure Core 4

Revision Notes

**March 2016**



# Pure Core 4

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# 1 Algebra

## Partial fractions

- 1) You must start with a proper fraction: i.e. the degree of the numerator **must be less than** the degree of the denominator.

If this is not the case you must **first** do long division to find quotient and remainder.

- 2) (a) Linear factors (not repeated)

$$\frac{\dots\dots}{(ax-b)(\dots\dots)} \equiv \frac{A}{(ax-b)} + \dots\dots\dots$$

- (b) Linear repeated factors (squared)

$$\frac{\dots\dots}{(ax-b)^2(\dots\dots)} \equiv \frac{A}{(ax-b)^2} + \frac{B}{(ax-b)} + \dots\dots\dots$$

- (c) Quadratic factors)

$$\frac{\dots\dots}{(ax^2+b)(\dots\dots)} \equiv \frac{Ax+B}{(ax^2+b)} + \dots\dots\dots$$

or 
$$\frac{\dots\dots}{(ax^2+bx+c)(\dots\dots)} \equiv \frac{Ax+B}{(ax^2+bx+c)} + \dots\dots\dots$$

*Example:* Express  $\frac{5-x+2x^2}{(1-x)(1+x^2)}$  in partial fractions.

*Solution:* The degree of the numerator, 2, is less than the degree of the denominator, 3, so we do not need long division and can write

$$\frac{5-x+2x^2}{(1-x)(1+x^2)} \equiv \frac{A}{1-x} + \frac{Bx+C}{1+x^2} \quad \text{multiply both sides by } (1-x)(1+x^2)$$

$$\Rightarrow 5-x+2x^2 \equiv A(1+x^2) + (Bx+C)(1-x)$$

$$\Rightarrow 5-1+2 = 2A \quad \Rightarrow \quad A=3 \quad \text{clever value!, put } x=1$$

$$\Rightarrow 5 = A+C \quad \Rightarrow \quad C=2 \quad \text{easy value, put } x=0$$

$$\Rightarrow 2 = A-B \quad \Rightarrow \quad B=1 \quad \text{equate coefficients of } x^2$$

$$\Rightarrow \frac{5-x+2x^2}{(1-x)(1+x^2)} \equiv \frac{3}{1-x} + \frac{x+2}{1+x^2}$$

N.B. You can put in any value for  $x$ , so you can always find as many equations as you need to solve for  $A, B, C, D, \dots$

*Example:* Express  $\frac{x^2 - 7x + 22}{(2x - 1)(x - 3)^2}$  in partial fractions.

*Solution:* The degree of the numerator, 2, is less than the degree of the denominator, 3, so we do not need long division and can write

$$\frac{x^2 - 7x + 22}{(2x - 1)(x - 3)^2} \equiv \frac{A}{2x - 1} + \frac{B}{(x - 3)^2} + \frac{C}{x - 3} \quad \text{multiply by denominator}$$

$$\Rightarrow x^2 - 7x + 22 \equiv A(x - 3)^2 + B(2x - 1) + C(2x - 1)(x - 3)$$

$$\Rightarrow 9 - 21 + 22 = 5B \quad \Rightarrow \quad B = 2 \quad \text{clever value, put } x = 3$$

$$\Rightarrow \frac{1}{4} - \frac{7}{2} + 22 = 2.5^2 A \quad \Rightarrow \quad A = 3 \quad \text{clever value, put } x = \frac{1}{2}$$

$$\Rightarrow 22 = 9A - B + 3C \quad \Rightarrow \quad C = -1 \quad \text{easy value, put } x = 0$$

$$\Rightarrow \frac{x^2 - 7x + 22}{(2x - 1)(x - 3)^2} \equiv \frac{3}{2x - 1} + \frac{2}{(x - 3)^2} - \frac{1}{x - 3}$$

*Example:* Express  $\frac{x^3 + x^2 - 9x - 3}{x^2 - 9}$  in partial fractions.

*Solution:* Firstly the degree of the numerator is **not less** than the degree of the denominator so we must divide top by bottom.

$$\begin{array}{r} x^2 - 9 \ ) \ x^3 \ + \ x^2 \ - \ 9x \ - \ 3 \quad ( \ x + 1 \\ \underline{x^3 \phantom{+ x^2} - \ 9x} \phantom{ - 3} \\ \phantom{x^3} \ x^2 \phantom{ - 9x} \phantom{ - 3} \\ \underline{\phantom{x^3} x^2 \phantom{ - 9x} - \ 9} \\ \phantom{x^3} \phantom{x^2} \phantom{ - 9x} \ 6 \end{array}$$

$$\Rightarrow \frac{x^3 + x^2 - 9x - 3}{x^2 - 9} = x + 1 + \frac{6}{x^2 - 9}$$

Factorise to give  $x^2 - 9 = (x - 3)(x + 3)$  and write

$$\frac{6}{x^2 - 9} \equiv \frac{6}{(x - 3)(x + 3)} \equiv \frac{A}{x - 3} + \frac{B}{x + 3} \quad \text{multiplying by denominator}$$

$$\Rightarrow 6 \equiv A(x + 3) + B(x - 3)$$

$$\Rightarrow 6 = 6A \quad \Rightarrow \quad A = 1 \quad \text{clever value, put } x = 3$$

$$\Rightarrow 6 = -6B \quad \Rightarrow \quad B = -1 \quad \text{clever value, put } x = -3$$

$$\Rightarrow \frac{x^3 + x^2 - 9x - 3}{x^2 - 9} = x + 1 + \frac{1}{x - 3} - \frac{1}{x + 3}$$

## 2 Coordinate Geometry

### Parametric equations

If we define  $x$  and  $y$  in terms of a single variable (the letters  $t$  or  $\theta$  are often used) then this variable is called a parameter: we then have the parametric equation of a curve.

*Example:*  $y = t^2 - 3$ ,  $x = 2 + t$  is the parametric equation of a curve. Find

- (i) the points where the curve meets the  $x$ -axis,
- (ii) the points of intersection of the curve with the line  $y = 2x + 1$ .

*Solution:*

- (i) The curve meets the  $x$ -axis when  $y = 0 \Rightarrow t^2 = 3 \Rightarrow t = \pm\sqrt{3}$   
 $\Rightarrow$  curve meets the  $x$ -axis at  $(2 - \sqrt{3}, 0)$  and  $(2 + \sqrt{3}, 0)$ .
- (ii) Substitute for  $x$  and  $y$  in the equation of the line  
 $\Rightarrow t^2 - 3 = 2(2 + t) + 1$   
 $\Rightarrow t^2 - 2t - 8 = 0 \Rightarrow (t - 4)(t + 2) = 0$   
 $\Rightarrow t = -2$  or  $4$   
 $\Rightarrow$  the points of intersection are  $(0, 1)$  and  $(6, 13)$ .

*Example:* Find whether the curves  $x = 2t + 3$ ,  $y = t^2 - 2$  and  $x = s - 1$ ,  $y = s - 3$  intersect. If they do give the point of intersection, otherwise give reasons why they do not intersect.

*Solution:* If they intersect there must be values of  $t$  and  $s$  (not necessarily the same), which make their  $x$ -coordinates equal, so find these values of  $t$  and  $s$

$$\Rightarrow 2t + 3 = s - 1 \Rightarrow s = 2t + 4$$

The  $y$ -coordinates must also be equal for the same values of  $t$  and  $s$

$$\Rightarrow t^2 - 2 = s - 3 = (2t + 4) - 3 = 2t + 1$$

$$\Rightarrow t^2 - 2t - 3 = 0 \Rightarrow (t - 3)(t + 1) = 0$$

$$\Rightarrow t = 3, s = 10 \text{ or } t = -1, s = 2$$

$\Rightarrow$  Curves intersect at  $t = 3$  giving  $(9, 7)$ . Check  $s = 10$  gives  $(9, 7)$ .

Or curves intersect at  $t = -1$  giving  $(1, -1)$ . Check  $s = 2$  giving  $(1, -1)$ .

## Conversion from parametric to Cartesian form

Eliminate the parameter ( $t$  or  $\theta$  or ...) to form an equation between  $x$  and  $y$  **only**.

*Example:* Find the Cartesian equation of the curve given by  $y = t^2 - 3$ ,  $x = t + 2$ .

*Solution:*  $x = t + 2 \Rightarrow t = x - 2$ , and  $y = t^2 - 3$

$$\Rightarrow y = (x - 2)^2 - 3,$$

which is the Cartesian equation of a parabola with vertex at  $(2, -3)$

With trigonometric parametric equations the formulae

$$\sin^2 t + \cos^2 t = 1 \quad \text{and} \quad \sec^2 t - \tan^2 t = 1$$

will often be useful.

*Example:* Find the Cartesian equation of the curve given by

$$y = 3\sin t + 2, \quad x = 3\cos t - 1.$$

*Solution:* Re-arranging we have

$$\sin t = \frac{y-2}{3}, \quad \text{and} \quad \cos t = \frac{x+1}{3}, \quad \text{which together with } \sin^2 t + \cos^2 t = 1$$

$$\Rightarrow \left(\frac{y-2}{3}\right)^2 + \left(\frac{x+1}{3}\right)^2 = 1$$

$$\Rightarrow (x+1)^2 + (y-2)^2 = 9$$

which is the Cartesian equation of a circle with centre  $(-1, 2)$  and radius 3.

*Example:* Find the Cartesian equation of the curve given by  $y = 3\tan t$ ,  $x = 4\sec t$ .

Hence sketch the curve.

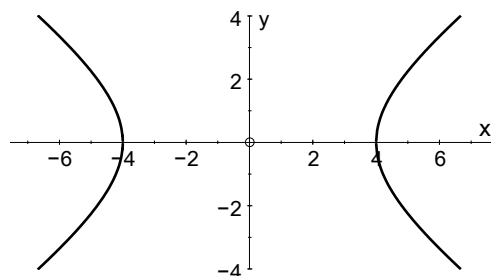
*Solution:* Re-arranging we have  $\tan t = \frac{y}{3}$ , and  $\sec t = \frac{x}{4}$ ,

which together with  $\sec^2 t - \tan^2 t = 1$

$$\Rightarrow \frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$

which is the standard equation of a hyperbola with centre  $(0, 0)$

and  $x$ -intercepts  $(4, 0)$ ,  $(-4, 0)$ .





## Area under curve given parametrically

We know that the area between a curve and the  $x$ -axis is given by  $A = \int y \, dx$

$$\Rightarrow \frac{dA}{dx} = y.$$

But, from the chain rule  $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} \Rightarrow \frac{dA}{dt} = y \frac{dx}{dt}$

Integrating with respect to  $t$

$$\Rightarrow A = \int y \frac{dx}{dt} dt.$$

*Example:* Find the area between the curve  $y = t^2 - 1$ ,  $x = t^3 + t$ , the  $x$ -axis and the lines  $x = 0$  and  $x = 2$ .

*Solution:* The area is  $A = \int_0^2 y \, dx$ .

$$\Rightarrow A = \int_?^? y \frac{dx}{dt} dt$$

we should write limits for  $t$ , **not**  $x$

**Firstly** we need to find  $y$  and  $\frac{dx}{dt}$  in terms of  $t$ .

$$y = t^2 - 1 \quad \text{and} \quad \frac{dx}{dt} = 3t^2 + 1.$$

**Secondly** we are integrating with respect to  $t$  and so the limits of integration must be for values of  $t$ .

$$x = 0 \Rightarrow t = 0, \text{ and}$$

$$x = 2 \Rightarrow t^3 + t = 2 \Rightarrow t^3 + t - 2 = 0 \Rightarrow (t - 1)(t^2 + t + 2) = 0 \Rightarrow t = 1.$$

so the limits for  $t$  are from 0 to 1

$$\Rightarrow A = \int_0^1 y \frac{dx}{dt} dt = \int_0^1 (t^2 - 1)(3t^2 + 1) dt$$

$$= \int_0^1 3t^4 - 2t^2 - 1 \, dt$$

$$= \left[ \frac{3t^5}{5} - \frac{2t^3}{3} - t \right]_0^1 = -1\frac{1}{15}$$

Note that in simple problems you may be able to eliminate  $t$  and find  $\int y \, dx$  in the usual manner. However there will be some problems where this is difficult and the above technique will be better.

### 3 Sequences and series

#### Binomial series $(1 + x)^n$ for any $n$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} \times x^2 + \frac{n(n-1)(n-2)}{3!} \times x^3 + \dots$$

This converges provided that  $|x| < 1$ .

*Example:* Expand  $(1 + 3x)^{-2}$ , giving the first four terms, and state the values of  $x$  for which the series is convergent.

*Solution:*

$$\begin{aligned} (1 + 3x)^{-2} &= 1 + (-2) \times 3x + \\ &\frac{(-2) \times (-3)}{2!} \times (3x)^2 + \frac{(-2) \times (-3) \times (-4)}{3!} \times (3x)^3 + \dots \\ &= 1 - 6x + 27x^2 - 108x^3 + \dots \end{aligned}$$

This series is convergent when  $|3x| < 1 \Leftrightarrow |x| < \frac{1}{3}$ .

*Example:* Use the previous example to find an approximation for  $\frac{1}{0.9997^2}$ .

*Solution:* Notice that  $\frac{1}{0.9997^2} = 0.9997^{-2} = (1 + 3x)^{-2}$  when  $x = -0.0001$ .

$$\begin{aligned} \text{So writing } x = -0.0001 \text{ in the expansion } 1 - 6x + 27x^2 - 108x^3 \\ 0.9997^{-2} \approx 1 + 0.0006 + 0.00000027 + 0.000000000108 = 1.000600270108 \end{aligned}$$

The correct answer to 13 decimal places is 1.0006002701080 not bad eh?

*Example:* Expand  $(4 - x)^{\frac{1}{2}}$ , giving all terms up to and including the term in  $x^3$ , and state for what values of  $x$  the series is convergent.

*Solution:* As the formula holds for  $(1 + x)^n$  we first re-write

$$\begin{aligned} (4 - x)^{\frac{1}{2}} &= 4^{\frac{1}{2}} \left(1 - \frac{x}{4}\right)^{\frac{1}{2}} = 2 \times \left(1 - \frac{x}{4}\right)^{\frac{1}{2}} && \text{and now we can use the formula} \\ &= 2 \times \left(1 + \frac{1}{2} \left(-\frac{x}{4}\right) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!} \times \left(-\frac{x}{4}\right)^2 + \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{3!} \times \left(-\frac{x}{4}\right)^3 + \dots\right) \\ &= 2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512}. \end{aligned}$$

This expansion converges for  $\left|\frac{x}{4}\right| < 1 \Leftrightarrow |x| < 4$ .

*Example:* Find the expansion of  $\frac{3x-1}{x^2-x-6}$  in ascending powers of  $x$  up to  $x^2$ .

*Solution:* First write in partial fractions

$$\Rightarrow \frac{3x-1}{x^2-x-6} = \frac{2}{x+3} + \frac{1}{x-2}, \text{ which must now be written as}$$

$$\frac{2}{3(1+\frac{x}{3})} - \frac{1}{2(1-\frac{x}{2})} = \frac{2}{3}(1+\frac{x}{3})^{-1} - \frac{1}{2}(1-\frac{x}{2})^{-1}$$

$$= \frac{2}{3} \left( 1 + (-1) \binom{x}{3} + \frac{(-1)(-2)}{2!} \binom{x}{3}^2 \right) - \frac{1}{2} \left( 1 + (-1) \binom{-x}{2} + \frac{(-1)(-2)}{2!} \binom{-x}{2}^2 \right)$$

$$= \frac{1}{6} - \frac{17x}{36} - \frac{11x^2}{216}.$$

## 4 Differentiation

### Relationship between $\frac{dy}{dx}$ and $\frac{dx}{dy}$

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\text{So } y = 3x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 6x \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{6x}.$$

### Implicit differentiation

This is just the chain rule when we do not know *explicitly* what  $y$  is as a function of  $x$ .

*Examples:* The following examples use the chain rule (or implicit differentiation)

$$\frac{d(y^3)}{dx} = \frac{d(y^3)}{dy} \times \frac{dy}{dx} = 3y^2 \frac{dy}{dx}$$

$$\frac{d(\sin y)}{dx} = \frac{d(\sin y)}{dy} \times \frac{dy}{dx} = \cos y \frac{dy}{dx}$$

$$\frac{d(5x^2y)}{dx} = 10xy + 5x^2 \frac{dy}{dx} \quad \text{using the product rule}$$

$$\frac{d}{dx}(x^2 + 3y)^3 = 3(x^2 + 3y)^2 \times \frac{d}{dx}(x^2 + 3y) = 3(x^2 + 3y)^2 \times \left(2x + 3\frac{dy}{dx}\right)$$

*Example:* Find the gradient of, and the equation of, the tangent to the curve

$$x^2 + y^2 - 3xy = -1 \quad \text{at the point } (1, 2).$$

*Solution:* Differentiating  $x^2 + y^2 - 3xy = -1$  with respect to  $x$  gives

$$2x + 2y \frac{dy}{dx} - \left(3y + 3x \frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{3y - 2x}{2y - 3x} \quad \Rightarrow \quad \frac{dy}{dx} = 4 \quad \text{when } x = 1 \text{ and } y = 2.$$

Equation of the tangent is  $y - 2 = 4(x - 1)$

$$\Rightarrow y = 4x - 2.$$

## Parametric differentiation

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

*Example:* A curve has parametric equations  $x = t^2 + t$ ,  $y = t^3 - 3t$ .

- (i) Find the equation of the normal at the point where  $t = 2$ .
- (ii) Find the points with zero gradient.

*Solution:* (i) When  $t = 2$ ,  $x = 6$  and  $y = 2$ .

$$\begin{aligned} \frac{dy}{dt} &= 3t^2 - 3 \text{ and } \frac{dx}{dt} = 2t + 1 \\ \Rightarrow \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t + 1} = \frac{9}{5} \text{ when } t = 2 \end{aligned}$$

Thus the gradient of the normal at the point  $(6, 2)$  is  $^{-5}/_9$   
and its equation is  $y - 2 = ^{-5}/_9(x - 6) \Rightarrow 5x + 9y = 48$ .

$$(ii) \text{ gradient} = 0 \text{ when } \frac{dy}{dx} = \frac{3t^2 - 3}{2t + 1} = 0$$

$$\Rightarrow 3t^2 - 3 = 0$$

$$\Rightarrow t = \pm 1$$

$\Rightarrow$  points with zero gradient are  $(0, 2)$  and  $(2, -2)$ .

## Exponential functions, $a^x$

$$y = a^x$$

$$\Rightarrow \ln y = \ln a^x = x \ln a$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \ln a \Rightarrow \frac{dy}{dx} = y \ln a$$

$$\Rightarrow \frac{d}{dx}(a^x) = a^x \ln a$$

## Related rates of change

We can use the chain rule to relate one rate of change to another.

*Example:* A spherical snowball is melting at a rate of  $96 \text{ cm}^3 \text{ s}^{-1}$  when its radius is  $12 \text{ cm}$ .

Find the rate at which its surface area is decreasing at that moment.

*Solution:* We know that  $V = \frac{4}{3} \pi r^3$  and that  $A = 4\pi r^2$ .

Using the chain rule we have

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times \frac{dr}{dt}, \quad \text{since } \frac{dV}{dr} = 4\pi r^2$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$$

$$\Rightarrow 96 = 4 \times \pi \times 12^2 \times \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{6\pi}$$

Using the chain rule again

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 8\pi r \times \frac{dr}{dt}, \quad \text{since } \frac{dA}{dr} = 8\pi r$$

$$\Rightarrow \frac{dA}{dt} = 8 \times \pi \times 12 \times \frac{1}{6\pi} = 16 \text{ cm}^2 \text{ s}^{-1}$$

## Forming differential equations

*Example:* The mass of a radio-active substance at time  $t$  is decaying at a rate which is proportional to the mass present at time  $t$ . Find a differential equation connecting the mass  $m$  and the time  $t$ .

*Solution:* Remember that  $\frac{dm}{dt}$  means the rate at which the mass is **increasing** so in this case we must consider the rate of decay as a negative increase

$$\Rightarrow \frac{dm}{dt} \propto -m$$

$$\Rightarrow \frac{dm}{dt} = -km, \text{ where } k \text{ is the (positive) constant of proportionality.}$$

$$\Rightarrow \int \frac{1}{m} dm = \int -kt dt$$

$$\Rightarrow \ln m = -\frac{1}{2}kt^2 + \ln A$$

$$\Rightarrow m = Ae^{-\frac{1}{2}kt^2}$$

## 5 Integration

### Integrals of $e^x$ and $\frac{1}{x}$

$$\int e^x dx = e^x + c$$

$$\int \frac{1}{x} dx = \ln|x| + c$$

for a further treatment of this result, see the appendix

*Example:* Find  $\int \frac{x^3 + 3x}{x^2} dx$

*Solution:*  $\int \frac{x^3 + 3x}{x^2} dx = \int x + \frac{3}{x} dx = \frac{1}{2}x^2 + 3 \ln x + c.$

### Standard integrals

**$x$  must be in RADIANS when integrating trigonometric functions.**

| $f(x)$        | $\int f(x) dx$        | $f(x)$                          | $\int f(x) dx$            |
|---------------|-----------------------|---------------------------------|---------------------------|
| $x^n$         | $\frac{x^{n+1}}{n+1}$ | $\sin x$                        | $-\cos x$                 |
| $\frac{1}{x}$ | $\ln x $              | $\cos x$                        | $\sin x$                  |
| $e^x$         | $e^x$                 | $\sec x \tan x$                 | $\sec x$                  |
|               |                       | $\sec^2 x$                      | $\tan x$                  |
|               |                       | $\operatorname{cosec} x \cot x$ | $-\operatorname{cosec} x$ |
|               |                       | $\operatorname{cosec}^2 x$      | $-\cot x$                 |

### Integration using trigonometric identities

*Example:* Find  $\int \cot^2 x dx.$

*Solution:*  $\cot^2 x = \operatorname{cosec}^2 x - 1$

$$\Rightarrow \int \cot^2 x dx = \int \operatorname{cosec}^2 x - 1 dx$$

$$= -\cot x - x + c.$$

*Example:* Find  $\int \sin^2 x \, dx$ .

*Solution:*  $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$   
 $\Rightarrow \int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx$   
 $= \frac{1}{2}x - \frac{1}{4}\sin 2x + c.$

You **cannot** change  $x$  to  $3x$  in the above result to find  $\int \sin^2 3x \, dx$ . see next example

*Example:* Find  $\int \sin^2 3x \, dx$ .

*Solution:*  $\sin^2 3x = \frac{1}{2}(1 - \cos 2 \times 3x) = \frac{1}{2}(1 - \cos 6x)$   
 $\Rightarrow \int \sin^2 3x \, dx = \int \frac{1}{2}(1 - \cos 6x) \, dx$   
 $= \frac{1}{2}x - \frac{1}{12}\sin 6x + c.$

*Example:* Find  $\int \sin 3x \cos 5x \, dx$ .

*Solution:* Using the formula  $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

**This formula is NOT in the formula booklet – you can use the formulae for  $\sin(A \pm B)$  and add them**

$$\begin{aligned} & \int \sin 3x \cos 5x \, dx \\ = & \frac{1}{2} \int \sin 8x + \sin(-2x) \, dx = \frac{1}{2} \int \sin 8x - \sin 2x \, dx \\ = & -\frac{1}{16} \cos 8x + \frac{1}{4} \cos 2x + c. \end{aligned}$$

## Integration by ‘reverse chain rule’

Some integrals which are not standard functions can be integrated by thinking of the chain rule for differentiation.

*Example:* Find  $\int \sin^4 3x \cos 3x \, dx$ .

*Solution:*  $\int \sin^4 3x \cos 3x \, dx$

If we think of  $u = \sin 3x$ , then the integrand looks like  $u^4 \frac{du}{dx}$  if we ignore the

constants, which would integrate to give  $\frac{1}{5}u^5$

so we differentiate  $u^5 = \sin^5 3x$

to give  $\frac{d}{dx}(\sin^5 3x) = 5(\sin^4 3x) \times 3 \cos 3x = 15 \sin^4 3x \cos 3x$

which is 15 times what we want and so

$$\int \sin^4 3x \cos 3x \, dx = \frac{1}{15} \sin^5 3x + c$$



*Example:* Find  $\int \frac{x}{(2x^2 - 3)} dx$

*Solution:*  $\int \frac{x}{(2x^2 - 3)} dx$

If we think of  $u = (2x^2 - 3)$ , then the integrand looks like  $\frac{1}{u} \frac{du}{dx}$  if we ignore the

constants, which would integrate to  $\ln |u|$

so we differentiate  $\ln |u| = \ln |2x^2 - 3|$

to give  $\frac{d}{dx}(\ln |2x^2 - 3|) = \frac{1}{2x^2 - 3} \times 4x = \frac{4x}{(2x^2 - 3)}$

which is 4 times what we want and so

$$\int \frac{x}{(2x^2 - 3)} dx = \frac{1}{4} \ln |2x^2 - 3| + c.$$

**In general**  $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$

### Integrals of $\tan x$ and $\cot x$

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx,$$

and we now have  $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$

$$\Rightarrow \int \tan x dx = -\ln |\cos x| + c$$

$$\Rightarrow \int \tan x dx = \ln |\sec x| + c$$

$\cot x$  can be integrated by a similar method to give

$$\int \cot x dx = \ln |\sin x| + c$$

### Integrals of $\sec x$ and $\operatorname{cosec} x$

$$\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

The top is now the derivative of the bottom

and we have  $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$

$$\Rightarrow \int \sec x dx = \ln |\sec x + \tan x| + c$$

and similarly

$$\int \operatorname{cosec} x dx = -\ln |\operatorname{cosec} x + \cot x| + c$$

## Integration using partial fractions

For use with algebraic fractions where the denominator factorises.

*Example:* Find  $\int \frac{6x}{x^2 + x - 2} dx$

*Solution:* First express  $\frac{6x}{x^2 + x - 2}$  in partial fractions.

$$\frac{6x}{x^2 + x - 2} \equiv \frac{6x}{(x-1)(x+2)} \equiv \frac{A}{x-1} + \frac{B}{x+2}$$

$$\Rightarrow 6x \equiv A(x+2) + B(x-1).$$

$$\text{put } x = 1 \quad \Rightarrow \quad A = 2,$$

$$\text{put } x = -2 \quad \Rightarrow \quad B = 4$$

$$\Rightarrow \int \frac{6x}{x^2 + x - 2} dx = \int \frac{2}{x-1} + \frac{4}{x+2} dx$$

$$= 2 \ln |x-1| + 4 \ln |x+2| + c.$$

## Integration by substitution, indefinite

- (i) Use the given substitution involving  $x$  and  $u$ , (or find a suitable substitution).
- (ii) Find **either**  $\frac{du}{dx}$  **or**  $\frac{dx}{du}$ , whichever is easier and re-arrange to find  $dx$  in terms of  $du$ , i.e.  $dx = \dots du$
- (iii) Use the substitution in (i) to make the integrand a function of  $u$ , and use your answer to (ii) to replace  $dx$  by  $\dots du$ .
- (iv) Simplify and integrate the function of  $u$ .
- (v) Use the substitution in (i) to write your answer in terms of  $x$ .

*Example:* Find  $\int x\sqrt{3x^2 - 5} dx$  using the substitution  $u = 3x^2 - 5$ .

*Solution:* (i)  $u = 3x^2 - 5$

(ii)  $\frac{du}{dx} = 6x \quad \Rightarrow \quad dx = \frac{du}{6x}$

(iii) We can see that there an  $x$  will cancel, and  $\sqrt{3x^2 - 5} = \sqrt{u}$

$$\int x\sqrt{3x^2 - 5} dx = \int x\sqrt{u} \frac{du}{6x} = \int \frac{\sqrt{u}}{6} du$$

(iv)  $= \frac{1}{6} \int u^{\frac{1}{2}} du = \frac{1}{6} \times \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$

(v)  $= \frac{(3x^2 - 5)^{\frac{3}{2}}}{9} + c$

*Example:* Find  $\int \frac{1}{1+x^2} dx$  using the substitution  $x = \tan u$ .

*Solution:* (i)  $x = \tan u$ .

$$(ii) \quad \frac{dx}{du} = \sec^2 u \quad \Rightarrow \quad dx = \sec^2 u \, du.$$

$$(iii) \quad \int \frac{1}{1+x^2} dx = \int \frac{1}{1+\tan^2 u} \sec^2 u \, du$$

$$(iv) \quad = \int \frac{\sec^2 u}{\sec^2 u} du \quad \text{since } 1 + \tan^2 u = \sec^2 u$$

$$= \int du = u + c$$

$$(v) \quad = \tan^{-1} x + c.$$

*Example:* Find  $\int \frac{3x}{\sqrt{x^2-4}} dx$  using the substitution  $u^2 = x^2 - 4$ .

*Solution:* (i)  $u^2 = x^2 - 4$ .

(ii) We know that  $\frac{d}{dx}(u^2) = 2u \frac{du}{dx}$  so differentiating gives

$$2u \frac{du}{dx} = 2x \quad \Rightarrow \quad dx = \frac{u}{x} du.$$

(iii) We can see that an  $x$  will cancel and  $\sqrt{x^2-4} = u$  so

$$\int \frac{3x}{\sqrt{x^2-4}} dx = \int \frac{3x}{u} \times \frac{u}{x} du$$

$$(iv) \quad = \int 3 \, du = 3u + c$$

$$(v) \quad = 3\sqrt{x^2-4} + c$$

A justification of this technique is given in the appendix.

## Integration by substitution, definite

If the integral has limits then proceed as before but remember to change the limits from values of  $x$  to the corresponding values of  $u$ .

- Add (ii) (a) Change limits from  $x$  to  $u$ , and  
 new (v) Put in limits for  $u$ .

*Example:* Find  $\int_2^6 x\sqrt{3x-2} dx$  using the substitution  $u = 3x - 2$ .

*Solution:* (i)  $u = 3x - 2$ .

$$(ii) \quad \frac{du}{dx} = 3 \Rightarrow dx = \frac{du}{3}$$

(ii) (a) Change limits from  $x$  to  $u$

$$x = 2 \Rightarrow u = 3 \times 2 - 2 = 4, \text{ and } x = 6 \Rightarrow u = 3 \times 6 - 2 = 16$$

$$(iii) \quad \int_2^6 x\sqrt{3x-2} dx = \int_4^{16} \frac{u+2}{3} \times u^{\frac{1}{2}} \frac{du}{3}$$

$$(iv) \quad = \frac{1}{9} \int_4^{16} u^{\frac{3}{2}} + 2u^{\frac{1}{2}} du$$

$$= \frac{1}{9} \left[ \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + 2 \times \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_4^{16}$$

$$(v) \quad = \frac{1}{9} \left[ \frac{2}{5} \times 1024 + \frac{4}{3} \times 64 \right] - \frac{1}{9} \left[ \frac{2}{5} \times 32 + \frac{4}{3} \times 8 \right] = 52.4 \text{ to 3 S.F.}$$

## Choosing the substitution

In general put  $u$  equal to the ‘awkward bit’ – but there are some special cases where this will not help.

$$\int x^3(x^2 + 1)^5 dx \quad \text{put } u = x^2 + 1$$

$$\int \frac{3x}{(x-2)^2} dx \quad \text{put } u = x-2$$

$$\int x\sqrt{2x+5} dx \quad \text{put } u = 2x+5 \quad \text{or} \quad u^2 = 2x+5$$

$$\int x^n \sqrt{4-x^2} dx \quad \text{put } u \text{ or } u^2 = 4-x^2 \quad \text{only if } n \text{ is ODD}$$

put  $x = 2 \sin u$  **only if  $n$  is EVEN (or zero)**

$$\text{this makes } \sqrt{4-x^2} = \sqrt{4 \cos^2 u} = 2 \cos u$$

There are many more possibilities – use your imagination!!

## Integration by parts

The product rule for differentiation is

$$\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} \quad \Rightarrow \quad u \frac{dv}{dx} = \frac{d(uv)}{dx} - v \frac{du}{dx}$$
$$\Rightarrow \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

To integrate by parts

- (i) choose  $u$  and  $\frac{dv}{dx}$
- (ii) find  $v$  and  $\frac{du}{dx}$
- (iii) substitute in formula and integrate.

*Example:* Find  $\int x \sin x dx$

*Solution:* (i) Choose  $u = x$ , because it disappears when differentiated

and choose  $\frac{dv}{dx} = \sin x$

(ii)  $u = x \Rightarrow \frac{du}{dx} = 1$  and

$$\frac{dv}{dx} = \sin x \Rightarrow v = \int \sin x dx = -\cos x$$

(iii)  $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

$$\Rightarrow \int x \sin x dx = -x \cos x - \int 1 \times (-\cos x) dx$$

$$= -x \cos x + \sin x + c.$$

*Example:* Find  $\int \ln x dx$

*Solution:* (i) It does not look like a product,  $u \frac{dv}{dx}$ , but if we take  $u = \ln x$  and

$$\frac{dv}{dx} = 1 \text{ then } u \frac{dv}{dx} = \ln x \times 1 = \ln x$$

(ii)  $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$  and  $\frac{dv}{dx} = 1 \Rightarrow v = x$

(iii)  $\int \ln x \times 1 dx = x \ln x - \int x \times \frac{1}{x} dx$

$$= x \ln x - x + c.$$

## Area under curve

We found in Core 2 that the area under the curve is written as the integral  $\int_a^b y \, dx$ .

We can consider the area as approximately the sum of the rectangles shown.

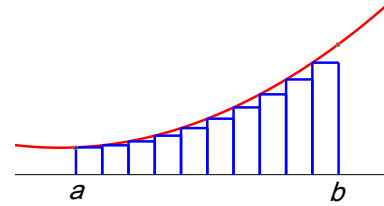
If each rectangle has width  $\delta x$  and if the heights of the rectangles are  $y_1, y_2, \dots, y_n$

then the area of the rectangles is approximately the area under the curve

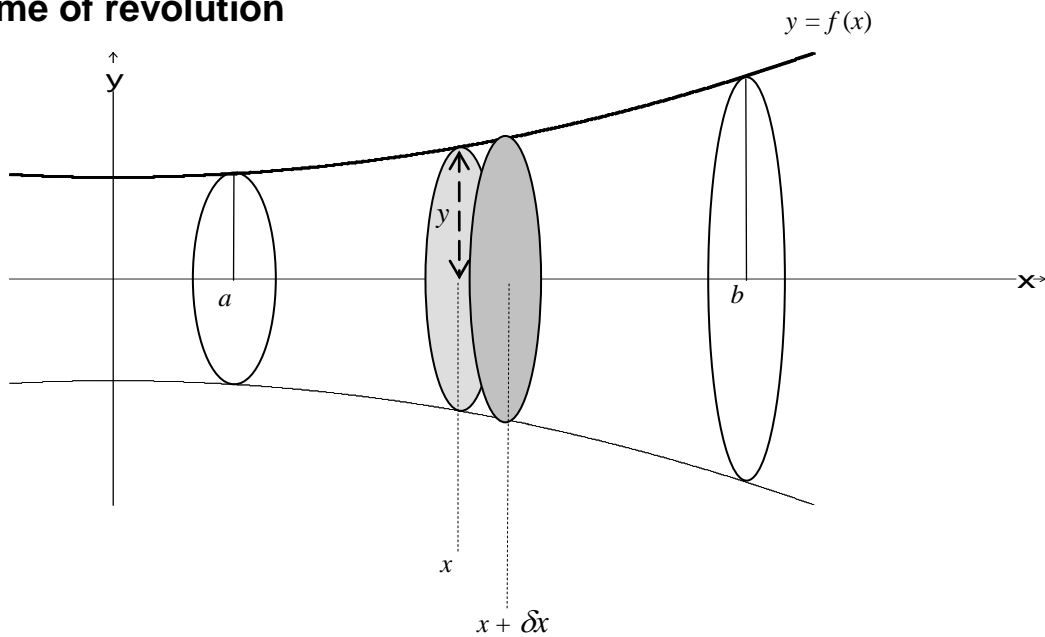
$$\sum_a^b y \delta x \cong \int_a^b y \, dx$$

and as  $\delta x \rightarrow 0$  we have  $\sum_a^b y \delta x \rightarrow \int_a^b y \, dx$

This last result is true for any function  $y$ .



## Volume of revolution



If the curve of  $y = f(x)$  is rotated about the  $x$ -axis then the volume of the shape formed can be found by considering many slices each of width  $\delta x$ : one slice is shown.

The volume of this slice (a disc) is approximately  $\pi y^2 \delta x$

$\Rightarrow$  Sum of volumes of all slices from  $a$  to  $b \approx \sum_a^b \pi y^2 \delta x$

and as  $\delta x \rightarrow 0$  we have (using the result above  $\sum_a^b y \delta x \rightarrow \int_a^b y \, dx$ )

$$\sum_a^b \pi y^2 \delta x \rightarrow \int_a^b \pi y^2 \, dx .$$

### Volume of revolution about the x-axis

Volume when  $y=f(x)$ , between  $x=a$  and  $x=b$ , is rotated about the  $x$ -axis

$$\text{is } V = \int_a^b \pi y^2 dx.$$

### Volume of revolution about the y-axis

Volume when  $y=f(x)$ , between  $y=c$  and  $y=d$ , is rotated about the  $y$ -axis

$$\text{is } V = \int_c^d \pi x^2 dy.$$

Volume of rotation about the  $y$ -axis is not in the syllabus but is included for completeness.

### Parametric integration

When  $x$  and  $y$  are given in parametric form we can find integrals using the techniques in integration by substitution.

$$\int y dx = \int y \frac{dx}{dt} dt \quad \text{think of 'cancelling' the 'dt's}$$

See the appendix for a justification of this result.

*Example:* If  $x = \tan t$  and  $y = \sin t$ , find the area under the curve from  $t=0$  to  $t = \pi/4$ .

*Solution:* The area =  $\int y dx$  for some limits on  $x = \int y \frac{dx}{dt} dt$ .

We know that  $y = \sin t$ , and also that

$$x = \tan t \Rightarrow \frac{dx}{dt} = \sec^2 t$$

$$\Rightarrow \text{area} = \int y dx = \int y \frac{dx}{dt} dt$$

$$= \int_0^{\pi/4} \sin t \sec^2 t dt = \int_0^{\pi/4} \tan t \sec t dt$$

$$= [\sec t]_0^{\pi/4} = \sqrt{2} - 1$$

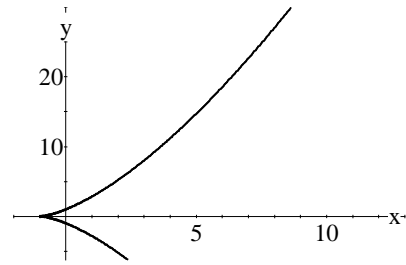
**To find a volume of revolution** we need  $\int \pi y^2 dx$  and we proceed as above writing

$$\int \pi y^2 dx = \int \pi y^2 \frac{dx}{dt} dt$$

*Example:* The curve shown has parametric equations

$$x = t^2 - 1, \quad y = t^3.$$

The section between  $x = 0$  and  $x = 8$  above the  $x$ -axis is rotated about the  $x$ -axis through  $2\pi$  radians. Find the volume generated.



*Solution:*  $V = \int_0^8 \pi y^2 dx.$

$$x = 0 \Rightarrow t = \pm 1 \text{ and } x = 8 \Rightarrow t = 3,$$

$$\text{but the curve is above the } x\text{-axis} \Rightarrow y = t^3 > 0 \Rightarrow t > 0, \Rightarrow t = +1, \text{ or } 3$$

$$\text{also } y = t^3, \quad x = t^2 - 1 \Rightarrow \frac{dx}{dt} = 2t$$

$$\Rightarrow V = \int \pi y^2 \frac{dx}{dt} dt = \int_1^3 \pi t^6 2t dt = 2\pi \int_1^3 t^7 dt$$

$$= 2\pi \left[ \frac{t^8}{8} \right]_1^3 = \frac{\pi}{4} (3^8 - 1)$$

## Differential equations

### Separating the variables

*Example:* Solve the differential equation  $\frac{dy}{dx} = 3y + xy.$

*Solution:*  $\frac{dy}{dx} = 3y + xy = y(3 + x)$

We first 'cheat' by separating the  $x$  s and  $y$  s onto different sides of the equation.

$$\Rightarrow \frac{1}{y} dy = (3 + x) dx \text{ and then put in the integral signs}$$

$$\Rightarrow \int \frac{1}{y} dy = \int (3 + x) dx \Rightarrow \ln y = 3x + \frac{1}{2} x^2 + c.$$

See the appendix for a justification of this technique.



## Exponential growth and decay

*Example:* A radio-active substance decays at a rate which is proportional to the mass of the substance present. Initially 25 grams are present and after 8 hours the mass has decreased to 20 grams. Find the mass after 1 day.

*Solution:* Let  $m$  grams be the mass of the substance at time  $t$ .

$\frac{dm}{dt}$  is the rate of **increase** of  $m$  so, since the mass is decreasing,

$$\frac{dm}{dt} = -km \quad \Rightarrow \quad \frac{1}{m} \frac{dm}{dt} = -k$$

$$\Rightarrow \int \frac{1}{m} dm = \int -k dt$$

$$\Rightarrow \ln |m| = -kt + \ln |A|$$

$$\Rightarrow \ln \left| \frac{m}{A} \right| = -kt$$

$$\Rightarrow m = Ae^{-kt}.$$

$$\text{When } t = 0, m = 25 \Rightarrow A = 25$$

$$\Rightarrow m = 25e^{-kt}.$$

$$\text{When } t = 8, m = 20$$

$$\Rightarrow 20 = 25e^{-8k} \quad \Rightarrow \quad e^{-8k} = 0.8$$

$$\Rightarrow -8k = \ln 0.8 \quad \Rightarrow \quad k = 0.027892943$$

$$\text{So when } t = 24, \quad m = 25e^{-24 \times 0.027892943} = 12.8.$$

Answer 12.8 grams after 1 day.

# 6 Vectors

## Notation

The book and exam papers like writing vectors in the form

$$\underline{a} = 3\underline{i} - 4\underline{j} + 7\underline{k}.$$

It is allowed, and sensible, to re-write vectors in column form

$$\text{i.e. } \underline{a} = 3\underline{i} - 4\underline{j} + 7\underline{k} = \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}.$$

## Definitions, adding and subtracting, etcetera

A vector has both magnitude (length) and direction. If you always think of a vector as a translation you will not go far wrong.

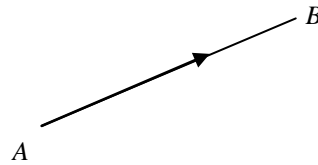
### Directed line segments

The vector  $\overrightarrow{AB}$  is the vector **from A to B**,

(or the translation which takes A to B).

This is sometimes called the

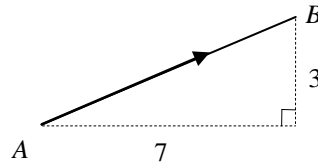
*displacement vector from A to B.*



### Vectors in co-ordinate form

Vectors can also be thought of as column vectors,

thus in the diagram  $\overrightarrow{AB} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ .



### Negative vectors

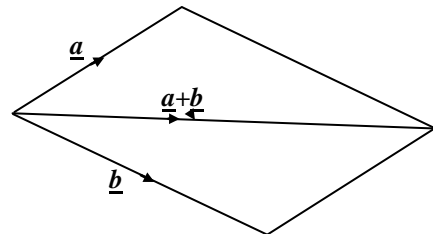
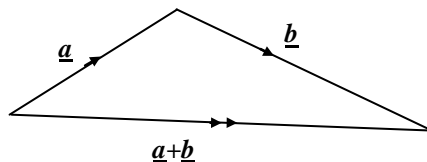
$\overrightarrow{BA}$  is the 'opposite' of  $\overrightarrow{AB}$  and so  $\overrightarrow{BA} = -\overrightarrow{AB} = \begin{bmatrix} -7 \\ -3 \end{bmatrix}$ .

### Adding and subtracting vectors

(i) Using a diagram

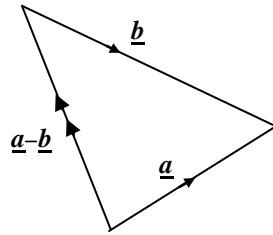
Geometrically this can be done using a triangle (or a parallelogram):

*Adding:*



The sum of two vectors is called the *resultant* of those vectors.

Subtracting:



(ii) Using coordinates

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a-c \\ b-d \end{bmatrix}.$$

## Parallel and non - parallel vectors

### Parallel vectors

Two vectors are parallel if they have the same direction

$\Leftrightarrow$  one is a multiple of the other.

*Example:* Which two of the following vectors are parallel?

$$\begin{bmatrix} 6 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

*Solution:* Notice that  $\begin{bmatrix} 6 \\ -3 \end{bmatrix} = \frac{-3}{2} \times \begin{bmatrix} -4 \\ 2 \end{bmatrix}$  and so  $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$  is parallel to  $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$

but  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is not a multiple of  $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$  and so cannot be parallel to the other two vectors.

*Example:* Find a vector of length 15 in the direction of  $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ .

*Solution:*  $\underline{\mathbf{a}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$  has length  $a = |\underline{\mathbf{a}}| = \sqrt{4^2 + 3^2} = 5$

and so the required vector of length  $15 = 3 \times 5$  is  $3\underline{\mathbf{a}} = 3 \times \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 12 \\ -9 \end{bmatrix}$ .

## Non-parallel vectors

If  $\underline{a}$  and  $\underline{b}$  are not parallel and if  $\alpha\underline{a} + \beta\underline{b} = \gamma\underline{a} + \delta\underline{b}$ , then

$$\alpha\underline{a} - \gamma\underline{a} = \delta\underline{b} - \beta\underline{b} \Rightarrow (\alpha - \gamma)\underline{a} = (\delta - \beta)\underline{b}$$

but  $\underline{a}$  and  $\underline{b}$  are not parallel and one cannot be a multiple of the other

$$\Rightarrow (\alpha - \gamma) = 0 = (\delta - \beta)$$

$$\Rightarrow \alpha = \gamma \quad \text{and} \quad \delta = \beta.$$

*Example:* If  $\underline{a}$  and  $\underline{b}$  are not parallel and if

$$\underline{b} + 2\underline{a} + \beta\underline{b} = \alpha\underline{a} + 3\underline{b} - 5\underline{a}, \text{ find the values of } \alpha \text{ and } \beta.$$

*Solution:* Since  $\underline{a}$  and  $\underline{b}$  are not parallel, the coefficients of  $\underline{a}$  and  $\underline{b}$  must 'balance out'

$$\Rightarrow 2 = \alpha - 5 \Rightarrow \alpha = 7 \quad \text{and} \quad 1 + \beta = 3 \Rightarrow \beta = 2.$$

## Modulus of a vector and unit vectors

### Modulus

The *modulus* of a vector is its magnitude or length.

If  $\overrightarrow{AB} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$  then the modulus of  $\overrightarrow{AB}$  is  $AB = |\overrightarrow{AB}| = \sqrt{7^2 + 3^2} = \sqrt{58}$

Or, if  $\underline{c} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$  then the modulus of  $\underline{c}$  is  $c = |\underline{c}| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$

### Unit vectors

A *unit vector* is one with length 1.

*Example:* Find a unit vector in the direction of  $\begin{bmatrix} -12 \\ 5 \end{bmatrix}$ .

*Solution:*  $\underline{a} = \begin{bmatrix} -12 \\ 5 \end{bmatrix}$  has length  $|\underline{a}| = a = \sqrt{12^2 + 5^2} = 13,$

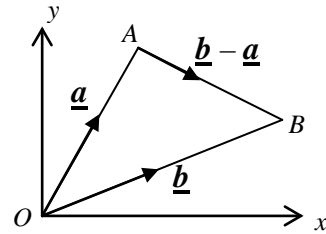
and so the required *unit* vector is  $\frac{1}{13} \times \underline{a} = \frac{1}{13} \times \begin{bmatrix} -12 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{-12}{13} \\ \frac{5}{13} \end{bmatrix}.$

## Position vectors

If  $A$  is the point  $(-1, 4)$  then the position vector of  $A$  is the vector from the origin to  $A$ , usually written as  $\overrightarrow{OA} = \underline{a} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ .

For two points  $A$  and  $B$  the position vectors are  $\overrightarrow{OA} = \underline{a}$  and  $\overrightarrow{OB} = \underline{b}$

To find the vector  $\overrightarrow{AB}$  go from  $A \rightarrow O \rightarrow B$   
giving  $\overrightarrow{AB} = -\underline{a} + \underline{b} = \underline{b} - \underline{a}$



## Ratios

*Example:*  $A, B$  are the points  $(2, 1)$  and  $(4, 7)$ .  $M$  lies on  $AB$  in the ratio  $1 : 3$ . Find the coordinates of  $M$ .

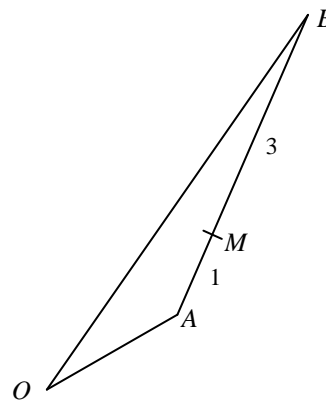
*Solution :*  $\overrightarrow{AB} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$$\overrightarrow{AM} = \frac{1}{4}\overrightarrow{AB} = \frac{1}{4} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$

$$\Rightarrow \overrightarrow{OM} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$$

$$\Rightarrow M \text{ is } (2.5, 2.5)$$



## Proving geometrical theorems

*Example:* In a triangle  $OBC$  let  $M$  and  $N$  be the midpoints of  $OB$  and  $OC$ .

Prove that  $BC = 2MN$  and that  $BC$  is parallel to  $MN$ .

*Solution:* Write the vectors  $\overrightarrow{OB}$  as  $\underline{b}$ , and  $\overrightarrow{OC}$  as  $\underline{c}$ .

$$\text{Then } \overrightarrow{OM} = \frac{1}{2} \overrightarrow{OB} = \frac{1}{2} \underline{b}$$

$$\text{and } \overrightarrow{ON} = \frac{1}{2} \overrightarrow{OC} = \frac{1}{2} \underline{c}.$$

To find  $\overrightarrow{MN}$ , go from  $M$  to  $O$  using  $-\frac{1}{2} \underline{b}$  and then from  $O$  to  $N$  using  $\frac{1}{2} \underline{c}$

$$\Rightarrow \overrightarrow{MN} = -\frac{1}{2} \underline{b} + \frac{1}{2} \underline{c}$$

$$\Rightarrow \overrightarrow{MN} = \frac{1}{2} \underline{c} - \frac{1}{2} \underline{b}$$

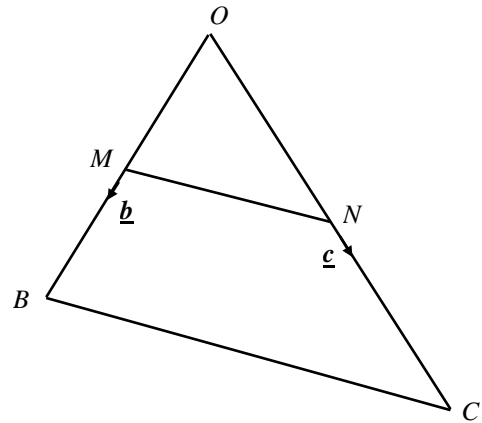
Also, to find  $\overrightarrow{BC}$ , go from  $B$  to  $O$  using  $-\underline{b}$  and then from  $O$  to  $C$  using  $\underline{c}$

$$\Rightarrow \overrightarrow{BC} = -\underline{b} + \underline{c} = \underline{c} - \underline{b}.$$

$$\text{But } \overrightarrow{MN} = -\frac{1}{2} \underline{b} + \frac{1}{2} \underline{c} = \frac{1}{2} (\underline{c} - \underline{b}) = \frac{1}{2} \overrightarrow{BC}$$

$\Rightarrow BC$  is parallel to  $MN$

and  $BC$  is twice as long as  $MN$ .



*Example:*  $P$  lies on  $OA$  in the ratio  $2 : 1$ , and  $Q$  lies on  $OB$  in the ratio  $2 : 1$ . Prove that  $PQ$  is parallel to  $AB$  and that  $PQ = \frac{2}{3}AB$ .

*Solution:* Let  $\underline{a} = \overrightarrow{OA}$ , and  $\underline{b} = \overrightarrow{OB}$

$$\Rightarrow \overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB} = \underline{b} - \underline{a}$$

$$\overrightarrow{OP} = \frac{2}{3} \overrightarrow{OA} = \frac{2}{3} \underline{a}, \quad \overrightarrow{OQ} = \frac{2}{3} \overrightarrow{OB} = \frac{2}{3} \underline{b}$$

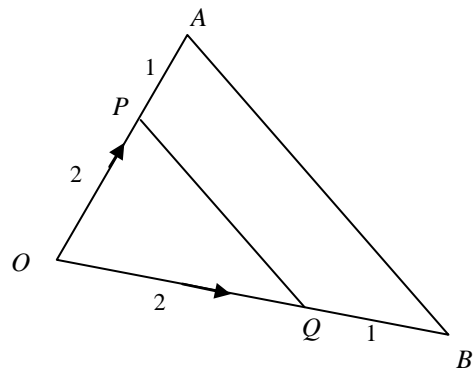
$$\text{and } \overrightarrow{PQ} = -\overrightarrow{OP} + \overrightarrow{OQ} = \frac{2}{3} \underline{b} - \frac{2}{3} \underline{a}$$

$$= \frac{2}{3} (\underline{b} - \underline{a})$$

$$= \frac{2}{3} \overrightarrow{AB}$$

$\Rightarrow PQ$  is parallel to  $AB$  and

$\Rightarrow PQ = \frac{2}{3}AB$ .



## Three dimensional vectors

### Length, modulus or magnitude of a vector

The length, modulus or magnitude of the vector  $\overrightarrow{OA} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$  is

$$|\overrightarrow{OA}| = |\underline{a}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2},$$

a sort of three dimensional Pythagoras.

### Distance between two points

To find the distance between  $A, (a_1, a_2, a_3)$  and  $B, (b_1, b_2, b_3)$  we need to find the length of the vector  $\overrightarrow{AB}$ .

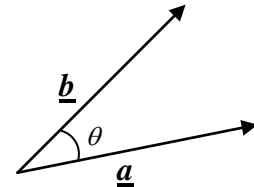
$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{bmatrix}$$

$$\Rightarrow |\overrightarrow{AB}| = AB = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$

### Scalar product

$$\underline{a} \cdot \underline{b} = ab \cos \theta$$

where  $a$  and  $b$  are the lengths of  $\underline{a}$  and  $\underline{b}$   
and  $\theta$  is the angle measured from  $\underline{a}$  to  $\underline{b}$ .



- Note that**
- (i)  $\underline{a} \cdot \underline{a} = aa \cos 0^\circ = a^2$
  - (ii)  $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$ .
  - (iii)  $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$

since  $\cos \theta = \cos (-\theta)$

### In co-ordinate form

$$\underline{a} \cdot \underline{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1 b_1 + a_2 b_2 = ab \cos \theta$$

$$\text{or } \underline{a} \cdot \underline{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 = ab \cos \theta.$$

## Perpendicular vectors

If  $\underline{a}$  and  $\underline{b}$  are perpendicular then  $\theta = 90^\circ$  and  $\cos \theta = 0$

thus  $\underline{a}$  perpendicular to  $\underline{b} \Rightarrow \underline{a} \cdot \underline{b} = 0$

and  $\underline{a} \cdot \underline{b} = 0 \Rightarrow$  either  $\underline{a}$  is perpendicular to  $\underline{b}$  or  $\underline{a} = \underline{0}$  or  $\underline{b} = \underline{0}$ .

*Example:* Find the values of  $\lambda$  so that  $\underline{a} = 3\underline{i} - 2\lambda\underline{j} + 2\underline{k}$  and  $\underline{b} = 2\underline{i} + \lambda\underline{j} + 6\underline{k}$  are perpendicular.

*Solution:* Since  $\underline{a}$  and  $\underline{b}$  are perpendicular  $\underline{a} \cdot \underline{b} = 0$

$$\Rightarrow \begin{bmatrix} 3 \\ -2\lambda \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ \lambda \\ 6 \end{bmatrix} = 0 \quad \Rightarrow \quad 6 - 2\lambda^2 + 12 = 0$$

$$\Rightarrow \lambda^2 = 9 \quad \Rightarrow \quad \lambda = \pm 3.$$

*Example:* Find a vector which is perpendicular to  $\underline{a}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ , and  $\underline{b}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ .

*Solution:* Let the vector  $\underline{c}, \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ , be perpendicular to both  $\underline{a}$  and  $\underline{b}$ .

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} p \\ q \\ r \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow p - q + 2r = 0 \quad \text{and} \quad 3p + q + r = 0.$$

Adding these equations gives  $4p + 3r = 0$ .

Notice that there will never be a unique solution to these problems, so having eliminated one variable,  $q$ , we find  $p$  in terms of  $r$ , and then find  $q$  in terms of  $r$ .

$$\Rightarrow p = \frac{-3r}{4} \quad \Rightarrow \quad q = \frac{5r}{4}$$

$$\Rightarrow \underline{c} \text{ is any vector of the form } \begin{bmatrix} \frac{-3r}{4} \\ \frac{5r}{4} \\ r \end{bmatrix},$$

and we choose a sensible value of  $r = 4$  to give  $\begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix}$ .



## Angle between vectors

*Example:* Find the angle between the vectors

$$\overrightarrow{OA} = 4\underline{i} - 5\underline{j} + 2\underline{k} \quad \text{and} \quad \overrightarrow{OB} = -\underline{i} + 2\underline{j} - 3\underline{k}, \quad \text{to the nearest degree.}$$

*Solution:* First re-write as column vectors (if you want)

$$\underline{a} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$

$$a = |\underline{a}| = \sqrt{4^2 + 5^2 + 2^2} = \sqrt{45} = 3\sqrt{5}, \quad b = |\underline{b}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\text{and} \quad \underline{a} \cdot \underline{b} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = -4 - 10 - 6 = -20$$

$$\underline{a} \cdot \underline{b} = ab \cos \theta \quad \Rightarrow \quad -20 = 3\sqrt{5} \times \sqrt{14} \cos \theta$$

$$\Rightarrow \cos \theta = \frac{-20}{3\sqrt{70}} = -0.796819$$

$$\Rightarrow \theta = 143^\circ \quad \text{to the nearest degree.}$$

## Vector equation of a straight line

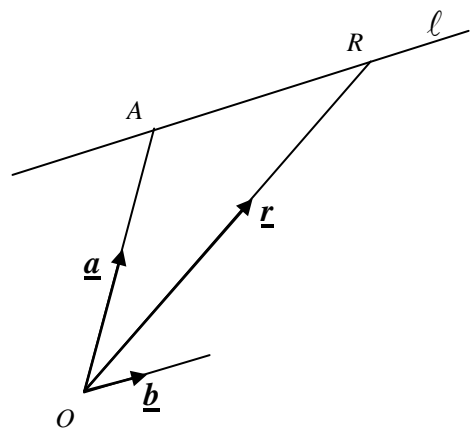
$\underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  is usually used as the position vector of a general point,  $R$ .

In the diagram the line  $\ell$  passes through the point  $A$  and is parallel to the vector  $\underline{b}$ .

To go from  $O$  to  $R$  first go to  $A$ , using  $\underline{a}$ , and then from  $A$  to  $R$  using some multiple of  $\underline{b}$ .

$\Rightarrow$  The equation of a straight line through the point  $A$  and parallel to the vector  $\underline{b}$  is

$$\underline{r} = \underline{a} + \lambda \underline{b}.$$



*Example:* Find the vector equation of the line through the points  $M$ ,  $(2, -1, 4)$ , and  $N$ ,  $(-5, 3, 7)$ .

*Solution:* We are looking for the line through  $M$  (or  $N$ ) which is parallel to the vector  $\overrightarrow{MN}$ .

$$\overrightarrow{MN} = \underline{n} - \underline{m} = \begin{bmatrix} -5 \\ 3 \\ 7 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \text{equation is } \underline{r} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}.$$

*Example:* Show that the point  $P$ ,  $(-1, 7, 10)$ , lies on the line

$$\underline{r} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

*Solution:* The  $x$  co-ord of  $P$  is  $-1$  and of the line is  $1 - \lambda$

$$\Rightarrow -1 = 1 - \lambda \Rightarrow \lambda = 2.$$

In the equation of the line this gives  $y = 7$  and  $z = 10$

$$\Rightarrow P, (-1, 7, 10) \text{ does lie on the line.}$$

## Intersection of two lines

### 2 Dimensions

*Example:* Find the intersection of the lines

$$\ell_1, \underline{r} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \ell_2, \underline{r} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

*Solution:* We are looking for values of  $\lambda$  and  $\mu$  which give the same  $x$  and  $y$  co-ordinates on each line.

$$\text{Equating } x \text{ co-ords} \Rightarrow 2 - \lambda = 1 + \mu$$

$$\text{equating } y \text{ co-ords} \Rightarrow \underline{3 + 2\lambda = 3 - \mu}$$

$$\text{Adding} \Rightarrow 5 + \lambda = 4 \Rightarrow \lambda = -1 \Rightarrow \mu = 2$$

$$\Rightarrow \text{lines intersect at } (3, 1).$$

### 3 Dimensions

This is similar to the method for 2 dimensions with one important difference – you can **not** be certain whether the lines intersect without checking.

You will always (or nearly always) be able to find values of  $\lambda$  and  $\mu$  by equating  $x$  coordinates and  $y$  coordinates but the  $z$  coordinates **might or might not be equal and must be checked**.

*Example:* Investigate whether the lines

$$\ell_1, \quad \underline{\mathbf{r}} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \ell_2, \quad \underline{\mathbf{r}} = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \text{intersect}$$

and if they do find their point of intersection.

*Solution:* **If** the lines intersect we can find values of  $\lambda$  and  $\mu$  to give the same  $x, y$  and  $z$  coordinates in each equation.

$$\begin{array}{llll} \text{Equating } x \text{ coords} & \Rightarrow & 2 - \lambda = -3 + \mu, & \mathbf{I} \\ \text{equating } y \text{ coords} & \Rightarrow & 1 + 2\lambda = 1 + 3\mu, & \mathbf{II} \\ \text{equating } z \text{ coords} & \Rightarrow & 3 + \lambda = 5 + \mu. & \mathbf{III} \end{array}$$

$$2 \times \mathbf{I} + \mathbf{II} \Rightarrow 5 = -5 + 5\mu \Rightarrow \mu = 2, \quad \text{in } \mathbf{I} \Rightarrow \lambda = 3.$$

**We must now check to see if we get the same point for the values of  $\lambda$  and  $\mu$**

In  $\ell_1$ ,  $\lambda = 3$  gives the point  $(-1, 7, 6)$ ;

in  $\ell_2$ ,  $\mu = 2$  gives the point  $(-1, 7, 7)$ .

The  $x$  and  $y$  co-ords are equal (as expected!), but the  $z$  co-ordinates are different and so the lines do **not** intersect.

## 7 Appendix

### Binomial series $(1 + x)^n$ for any $n$ – proof

Suppose that

$$f(x) = (1 + x)^n = a + bx + cx^2 + dx^3 + ex^4 + \dots$$

$$\text{put } x = 0, \Rightarrow 1 = a$$

$$\Rightarrow f'(x) = n(1 + x)^{n-1} = b + 2cx + 3dx^2 + 4ex^3 + \dots$$

$$\text{put } x = 0, \Rightarrow n = b$$

$$\Rightarrow f''(x) = n(n-1)(1 + x)^{n-2} = 2c + 3 \times 2dx + 4 \times 3ex^2 + \dots$$

$$\text{put } x = 0, \Rightarrow n(n-1) = 2c \Rightarrow \frac{n(n-1)}{2!} = c$$

$$\Rightarrow f'''(x) = n(n-1)(n-2)(1 + x)^{n-3} = 3 \times 2d + 4 \times 3 \times 2ex + \dots$$

$$\text{put } x = 0, \Rightarrow n(n-1)(n-2) = 3 \times 2d \Rightarrow \frac{n(n-1)(n-2)}{3!} = d$$

Continuing this process, we have

$$\frac{n(n-1)(n-2)(n-3)}{4!} = e \quad \text{and} \quad \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} = f, \quad \text{etc.}$$

giving  $f(x) = (1 + x)^n$

$$\begin{aligned} &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}x^5 + \dots \\ &+ \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots \end{aligned}$$

Showing that this is convergent for  $|x| < 1$ , is more difficult!

### Derivative of $x^q$ for $q$ rational

Suppose that  $q$  is any rational number,  $q = \frac{r}{s}$ , where  $r$  and  $s$  are integers,  $s \neq 0$ .

Then  $y = x^q = x^{r/s} \Rightarrow y^s = x^r$

Differentiating with respect to  $x \Rightarrow s \times y^{s-1} \frac{dy}{dx} = r \times x^{r-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{r}{s} \times \frac{x^{r-1}}{y^{s-1}} = \frac{r}{s} \times \frac{x^{r-1}}{y^s} \times y = q \times \frac{x^{r-1}}{y^s} \times y \quad \text{since } q = \frac{r}{s}$$

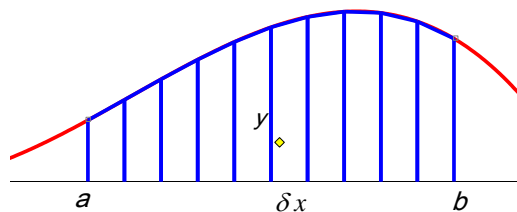
$$\Rightarrow \frac{dy}{dx} = q \times \frac{x^{r-1}}{x^r} \times x^q = qx^{q-1} \quad \text{since } y^s = x^r \text{ and } y = x^q$$

which follows the rule found for  $x^n$ , where  $n$  is an integer.

## $\int \frac{1}{x} dx$ for negative limits

We know that the 'area' under any curve, from  $x = a$  to  $x = b$  is approximately

$$\sum_a^b y \delta x \rightarrow \int_a^b y dx, \quad \text{as } \delta x \rightarrow 0$$



If the curve is above the  $x$ -axis, all the  $y$  values are positive, and if  $a < b$  then all values of  $\delta x$  are positive, and so the integral is positive.

$$\int_{-a}^{-b} \frac{1}{x} dx = \left[ \ln|x| \right]_{-a}^{-b} = \ln b - \ln a$$

*Example:* Find  $\int_{-1}^{-3} \frac{1}{x} dx$ .

*Solution:* The integral wanted is shown as  $A'$  in the diagram.

By symmetry  $|A'| = A$  ( $A$  positive)

and we need to decide whether the integral is  $+A$  or  $-A$ .

From  $x = -1$  to  $x = -3$ , we are going in the direction of  $x$  **decreasing**

$\Rightarrow$  all  $\delta x$  are negative.

And the graph is below the  $x$ -axis,

$\Rightarrow$  the  $y$  values are negative,  $\Rightarrow y \delta x$  is **positive**

$\Rightarrow \sum_{-1}^{-3} y \delta x > 0 = \int_{-1}^{-3} y dx > 0$

$\Rightarrow$  the integral is **positive** and equal to  $A$ .

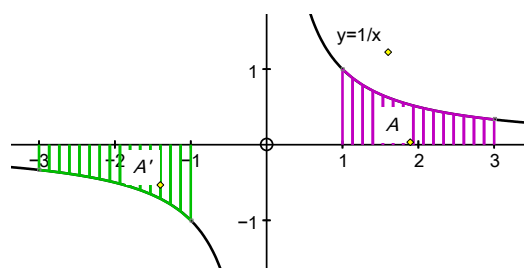
The integral,  $A' = A = \int_1^3 \frac{1}{x} dx = [\ln x]_1^3 = \ln 3 - \ln 1 = \ln 3$

$\Rightarrow A' = \int_{-1}^{-3} \frac{1}{x} dx = \ln 3$

Notice that this is what we get if we write  $\ln|x|$  in place of  $\ln x$

$$\int_{-1}^{-3} \frac{1}{x} dx = [\ln|x|]_{-1}^{-3} = \ln 3 - \ln 1 = \ln 3$$

As it will always be possible to use symmetry in this way, since we can never have one positive and one negative limit (because there is a discontinuity at  $x = 0$ ), it is correct to write  $\ln|x|$  for the integral of  $1/x$ .



## Integration by substitution – why it works

We show the general method with an example.

$$y = \int x^2 \sqrt{1+x^3} \, dx$$

$$\Rightarrow \frac{dy}{dx} = x^2 \sqrt{1+x^3} \quad \text{integrand} = x^2 \sqrt{1+x^3}$$

Choose  $u = 1+x^3$

$$\Rightarrow \frac{du}{dx} = 3x^2 \quad \Rightarrow \quad \frac{dx}{du} = \frac{1}{3x^2} \quad \text{rearrange to give } dx = \frac{du}{3x^2}$$

But  $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$

$$\Rightarrow y = \int \frac{dy}{dx} \times \frac{dx}{du} \, du$$

$$\frac{dy}{dx} = x^2 \sqrt{u} \quad \text{leave the } x^2 \text{ because it appears in } dx$$

$$y = \int x^2 \sqrt{u} \times \frac{1}{3x^2} \, du \quad \text{this is the same as writing the integrand in terms of } u, \text{ and then replacing } dx \text{ by } \frac{dx}{du} du = \frac{du}{3x^2}$$

The essential part of this method, writing the integrand in terms of  $u$ , and then replacing  $dx$  by  $\frac{dx}{du} du$ , will be the same for all integrations by substitution.

## Parametric integration

This is similar to integration by substitution.

$$A = \int y \, dx \quad \Rightarrow \quad \frac{dA}{dx} = y$$

$$\Rightarrow \frac{dA}{dt} \times \frac{dt}{dx} = y$$

$$\Rightarrow \frac{dA}{dt} = y \times \frac{dx}{dt}$$

$$\Rightarrow A = \int y \frac{dx}{dt} \, dt$$

## Separating the variables – why it works

We show this with an example.

$$\text{If } y = 6y^3 \text{ then } \frac{dy}{dx} = 18y^2 \frac{dy}{dx}$$

$$\text{and so } \int 18y^2 \frac{dy}{dx} dx = \int 18y^2 dy = 6y^3 + c$$

Notice that we ‘cancel’ the  $dx$ .

$$\text{Example: Solve } \frac{dy}{dx} = x^2 \sec y$$

$$\text{Solution: } \frac{dy}{dx} = x^2 \sec y$$

$$\Rightarrow \cos y \frac{dy}{dx} = x^2$$

$$\Rightarrow \int \cos y \frac{dy}{dx} dx = \int x^2 dx$$

$$\Rightarrow \int \cos y dy = \int x^2 dx$$

‘cancelling’ the  $dx$

$$\Rightarrow \sin y = \frac{1}{3}x^3 + c$$

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