The Grange School
Maths Department

Core 3
OCR Past Papers

1 The function f is defined for all real values of x by

$$f(x) = 10 - (x+3)^2.$$

(i) State the range of f. [1]

(ii) Find the value of $ff(-1)$. [3]

2 Find the exact solutions of the equation $|6x - 1| = |x - 1|$. [4]

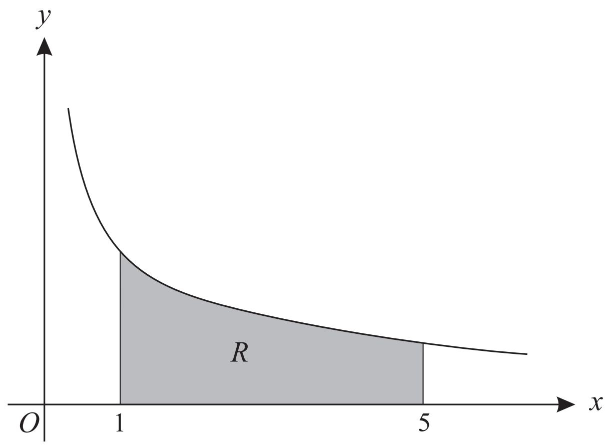
3 The mass, m grams, of a substance at time t years is given by the formula

$$m = 180e^{-0.017t}.$$

(i) Find the value of t for which the mass is 25 grams. [3]

(ii) Find the rate at which the mass is decreasing when $t = 55$. [3]

4 **(a)**



The diagram shows the curve $y = \dfrac{2}{\sqrt{x}}$. The region R, shaded in the diagram, is bounded by the curve and by the lines $x = 1$, $x = 5$ and $y = 0$. The region R is rotated completely about the x-axis. Find the exact volume of the solid formed. [4]

(b) Use Simpson's rule, with 4 strips, to find an approximate value for

$$\int_1^5 \sqrt{(x^2+1)}\,dx,$$

giving your answer correct to 3 decimal places. [4]

5 **(i)** Express $3\sin\theta + 2\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. [3]

(ii) Hence solve the equation $3\sin\theta + 2\cos\theta = \frac{7}{2}$, giving all solutions for which $0^\circ < \theta < 360^\circ$. [5]

June 2005

6 **(a)** Find the exact value of the x-coordinate of the stationary point of the curve $y = x \ln x$. [4]

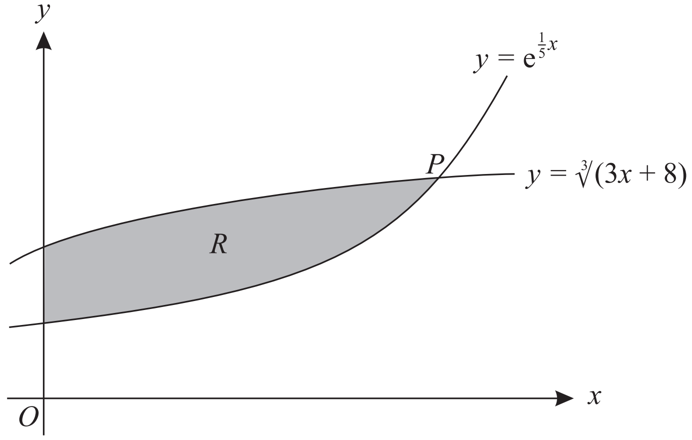
(b) The equation of a curve is $y = \dfrac{4x + c}{4x - c}$, where c is a non-zero constant. Show by differentiation that this curve has no stationary points. [3]

7 **(i)** Write down the formula for $\cos 2x$ in terms of $\cos x$. [1]

(ii) Prove the identity $\dfrac{4\cos 2x}{1 + \cos 2x} \equiv 4 - 2\sec^2 x$. [3]

(iii) Solve, for $0 < x < 2\pi$, the equation $\dfrac{4\cos 2x}{1 + \cos 2x} = 3\tan x - 7$. [5]

8



The diagram shows part of each of the curves $y = e^{\frac{1}{5}x}$ and $y = \sqrt[3]{(3x + 8)}$. The curves meet, as shown in the diagram, at the point P. The region R, shaded in the diagram, is bounded by the two curves and by the y-axis.

(i) Show by calculation that the x-coordinate of P lies between 5.2 and 5.3. [3]

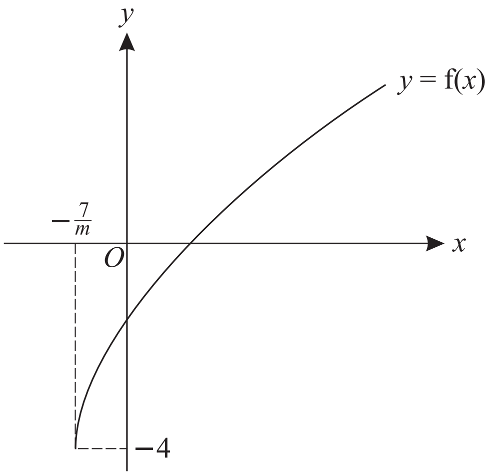
(ii) Show that the x-coordinate of P satisfies the equation $x = \frac{5}{3}\ln(3x + 8)$. [2]

(iii) Use an iterative formula, based on the equation in part **(ii)**, to find the x-coordinate of P correct to 2 decimal places. [3]

(iv) Use integration, and your answer to part **(iii)**, to find an approximate value of the area of the region R. [5]

[Question 9 is printed overleaf.]

9



The function f is defined by $f(x) = \sqrt{(mx + 7)} - 4$, where $x \geqslant -\dfrac{7}{m}$ and m is a positive constant. The diagram shows the curve $y = f(x)$.

(i) A sequence of transformations maps the curve $y = \sqrt{x}$ to the curve $y = f(x)$. Give details of these transformations. [4]

(ii) Explain how you can tell that f is a one-one function and find an expression for $f^{-1}(x)$. [4]

(iii) It is given that the curves $y = f(x)$ and $y = f^{-1}(x)$ do not meet. Explain how it can be deduced that neither curve meets the line $y = x$, and hence determine the set of possible values of m. [5]

Jan 2006

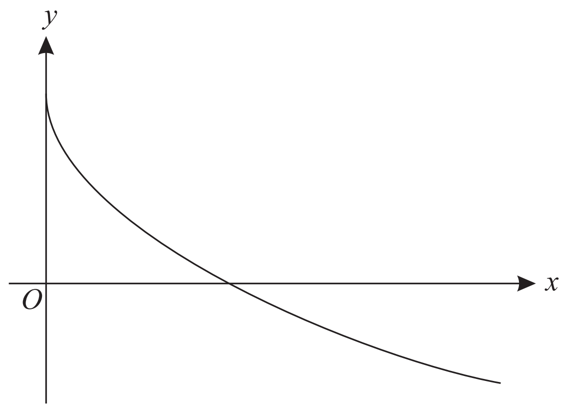
1 Show that $\int_2^8 \frac{3}{x}\,dx = \ln 64$. [4]

2 Solve, for $0° < \theta < 360°$, the equation $\sec^2\theta = 4\tan\theta - 2$. [5]

3 **(a)** Differentiate $x^2(x+1)^6$ with respect to x. [3]

(b) Find the gradient of the curve $y = \frac{x^2+3}{x^2-3}$ at the point where $x = 1$. [3]

4



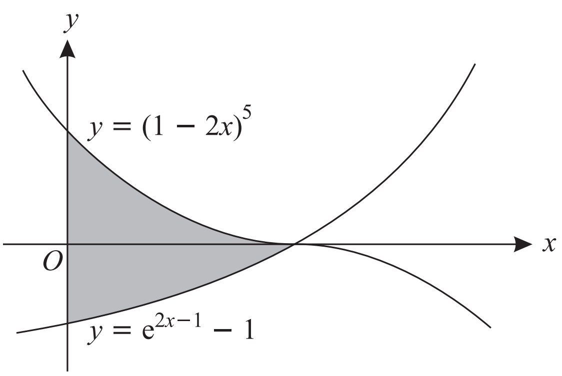
The function f is defined by $f(x) = 2 - \sqrt{x}$ for $x \geqslant 0$. The graph of $y = f(x)$ is shown above.

(i) State the range of f. [1]

(ii) Find the value of ff(4). [2]

(iii) Given that the equation $|f(x)| = k$ has two distinct roots, determine the possible values of the constant k. [2]

5



The diagram shows the curves $y = (1-2x)^5$ and $y = e^{2x-1} - 1$. The curves meet at the point $\left(\frac{1}{2}, 0\right)$. Find the exact area of the region (shaded in the diagram) bounded by the y-axis and by part of each curve. [8]

Jan 2006

6 **(a)**

t	0	10	20
X	275	440	

The quantity X is increasing exponentially with respect to time t. The table above shows values of X for different values of t. Find the value of X when $t = 20$. [3]

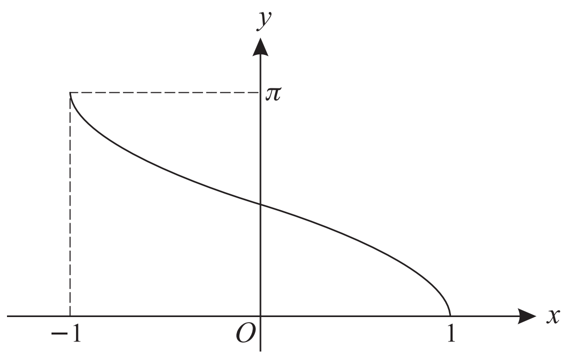
(b) The quantity Y is decreasing exponentially with respect to time t where

$$Y = 80e^{-0.02t}.$$

(i) Find the value of t for which $Y = 20$, giving your answer correct to 2 significant figures. [3]

(ii) Find by differentiation the rate at which Y is decreasing when $t = 30$, giving your answer correct to 2 significant figures. [3]

7



The diagram shows the curve with equation $y = \cos^{-1} x$.

(i) Sketch the curve with equation $y = 3\cos^{-1}(x - 1)$, showing the coordinates of the points where the curve meets the axes. [3]

(ii) By drawing an appropriate straight line on your sketch in part **(i)**, show that the equation $3\cos^{-1}(x - 1) = x$ has exactly one root. [1]

(iii) Show by calculation that the root of the equation $3\cos^{-1}(x - 1) = x$ lies between 1.8 and 1.9. [2]

(iv) The sequence defined by

$$x_1 = 2, \qquad x_{n+1} = 1 + \cos\left(\tfrac{1}{3}x_n\right)$$

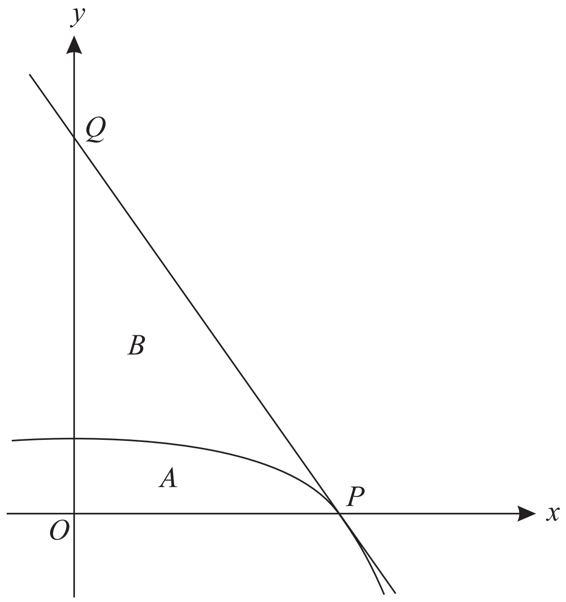
converges to a number α. Find the value of α correct to 2 decimal places and explain why α is the root of the equation $3\cos^{-1}(x - 1) = x$. [5]

[Questions 8 and 9 are printed overleaf.]

[Turn over

Jan 2006

8



The diagram shows part of the curve $y = \ln(5 - x^2)$ which meets the x-axis at the point P with coordinates (2, 0). The tangent to the curve at P meets the y-axis at the point Q. The region A is bounded by the curve and the lines $x = 0$ and $y = 0$. The region B is bounded by the curve and the lines PQ and $x = 0$.

(i) Find the equation of the tangent to the curve at P. [5]

(ii) Use Simpson's Rule with four strips to find an approximation to the area of the region A, giving your answer correct to 3 significant figures. [4]

(iii) Deduce an approximation to the area of the region B. [2]

9 **(i)** By first writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$ [4]

(ii) Determine the greatest possible value of

$$9\sin\left(\tfrac{10}{3}\alpha\right) - 12\sin^3\left(\tfrac{10}{3}\alpha\right),$$

and find the smallest positive value of α (in degrees) for which that greatest value occurs. [3]

(iii) Solve, for $0° < \beta < 90°$, the equation $3\sin 6\beta \operatorname{cosec} 2\beta = 4$. [6]

June 2006

1 Find the equation of the tangent to the curve $y = \sqrt{4x+1}$ at the point $(2, 3)$. [5]

2 Solve the inequality $|2x - 3| < |x + 1|$. [5]

3 The equation $2x^3 + 4x - 35 = 0$ has one real root.

(i) Show by calculation that this real root lies between 2 and 3. [3]

(ii) Use the iterative formula

$$x_{n+1} = \sqrt[3]{17.5 - 2x_n},$$

with a suitable starting value, to find the real root of the equation $2x^3 + 4x - 35 = 0$ correct to 2 decimal places. You should show the result of each iteration. [3]

4 It is given that $y = 5^{x-1}$.

(i) Show that $x = 1 + \dfrac{\ln y}{\ln 5}$. [2]

(ii) Find an expression for $\dfrac{dx}{dy}$ in terms of y. [2]

(iii) Hence find the exact value of the gradient of the curve $y = 5^{x-1}$ at the point $(3, 25)$. [2]

5 **(i)** Write down the identity expressing $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$. [1]

(ii) Given that $\sin\alpha = \frac{1}{4}$ and α is acute, show that $\sin 2\alpha = \frac{1}{8}\sqrt{15}$. [3]

(iii) Solve, for $0° < \beta < 90°$, the equation $5\sin 2\beta \sec\beta = 3$. [3]

June 2006

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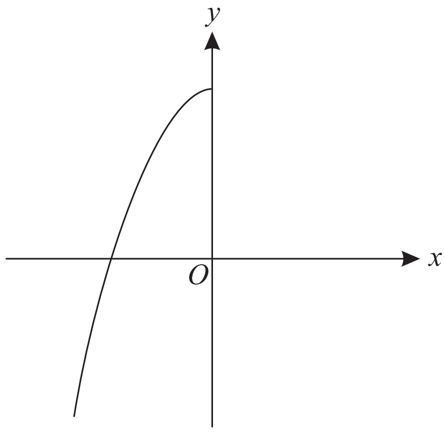


The diagram shows the graph of $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2 - x^2, \qquad x \leqslant 0.$$

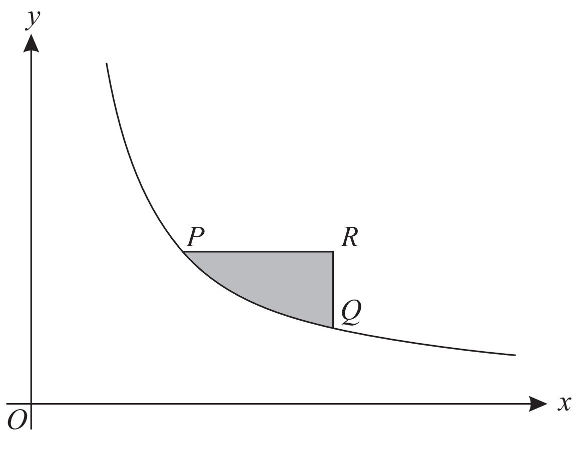
(i) Evaluate $\mathrm{ff}(-3)$. [3]

(ii) Find an expression for $\mathrm{f}^{-1}(x)$. [3]

(iii) Sketch the graph of $y = \mathrm{f}^{-1}(x)$. Indicate the coordinates of the points where the graph meets the axes. [3]

7 **(a)** Find the exact value of $\displaystyle\int_1^2 \frac{2}{(4x-1)^2}\,\mathrm{d}x$. [4]

(b)



The diagram shows part of the curve $y = \dfrac{1}{x}$. The point P has coordinates $\left(a, \dfrac{1}{a}\right)$ and the point Q has coordinates $\left(2a, \dfrac{1}{2a}\right)$, where a is a positive constant. The point R is such that PR is parallel to the x-axis and QR is parallel to the y-axis. The region shaded in the diagram is bounded by the curve and by the lines PR and QR. Show that the area of this shaded region is $\ln\left(\frac{1}{2}\mathrm{e}\right)$. [6]

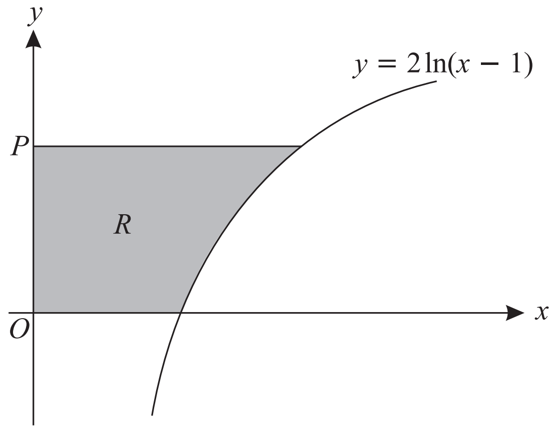
[Turn over

8 **(i)** Express $5\cos x + 12\sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [3]

(ii) Hence give details of a pair of transformations which transforms the curve $y = \cos x$ to the curve $y = 5\cos x + 12\sin x$. [3]

(iii) Solve, for $0° < x < 360°$, the equation $5\cos x + 12\sin x = 2$, giving your answers correct to the nearest 0.1°. [5]

9



The diagram shows the curve with equation $y = 2\ln(x - 1)$. The point P has coordinates $(0, p)$. The region R, shaded in the diagram, is bounded by the curve and the lines $x = 0$, $y = 0$ and $y = p$. The units on the axes are centimetres. The region R is rotated completely about the ***y*-axis** to form a solid.

(i) Show that the volume, V cm^3, of the solid is given by

$$V = \pi\left(e^p + 4e^{\frac{1}{2}p} + p - 5\right).$$ [8]

(ii) It is given that the point P is moving in the positive direction along the y-axis at a constant rate of 0.2 cm min^{-1}. Find the rate at which the volume of the solid is increasing at the instant when $p = 4$, giving your answer correct to 2 significant figures. [5]

Jan 2007

1 Find the equation of the tangent to the curve $y = \frac{2x+1}{3x-1}$ at the point $\left(1, \frac{3}{2}\right)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. [5]

2 It is given that θ is the acute angle such that $\sin\theta = \frac{12}{13}$. Find the exact value of

(i) $\cot\theta$, [2]

(ii) $\cos 2\theta$. [3]

3 **(a)** It is given that a and b are positive constants. By sketching graphs of

$$y = x^5 \quad \text{and} \quad y = a - bx$$

on the same diagram, show that the equation

$$x^5 + bx - a = 0$$

has exactly one real root. [3]

(b) Use the iterative formula $x_{n+1} = \sqrt[5]{53 - 2x_n}$, with a suitable starting value, to find the real root of the equation $x^5 + 2x - 53 = 0$. Show the result of each iteration, and give the root correct to 3 decimal places. [4]

4 **(i)** Given that $x = (4t+9)^{\frac{1}{2}}$ and $y = 6e^{\frac{1}{2}x+1}$, find expressions for $\frac{dx}{dt}$ and $\frac{dy}{dx}$. [4]

(ii) Hence find the value of $\frac{dy}{dt}$ when $t = 4$, giving your answer correct to 3 significant figures. [3]

5 **(i)** Express $4\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [3]

(ii) Hence solve the equation $4\cos\theta - \sin\theta = 2$, giving all solutions for which $-180° < \theta < 180°$. [5]

Jan 2007

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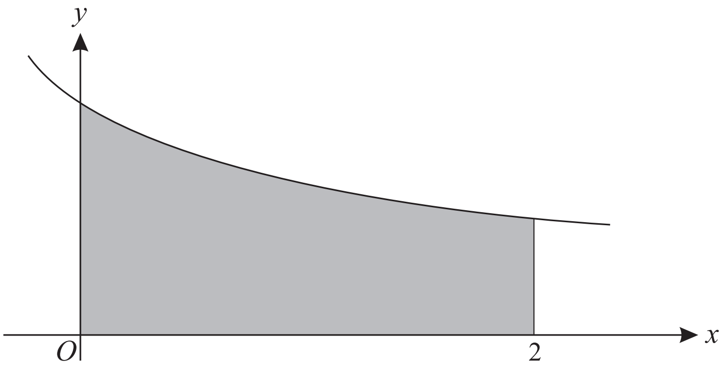

The diagram shows the curve with equation $y = \dfrac{1}{\sqrt{3x+2}}$. The shaded region is bounded by the curve and the lines $x = 0$, $x = 2$ and $y = 0$.

(i) Find the exact area of the shaded region. [4]

(ii) The shaded region is rotated completely about the x-axis. Find the exact volume of the solid formed, simplifying your answer. [5]

7 The curve $y = \ln x$ is transformed to the curve $y = \ln\left(\frac{1}{2}x - a\right)$ by means of a translation followed by a stretch. It is given that a is a positive constant.

(i) Give full details of the translation and stretch involved. [2]

(ii) Sketch the graph of $y = \ln\left(\frac{1}{2}x - a\right)$. [2]

(iii) Sketch, on another diagram, the graph of $y = \left|\ln\left(\frac{1}{2}x - a\right)\right|$. [2]

(iv) State, in terms of a, the set of values of x for which $\left|\ln\left(\frac{1}{2}x - a\right)\right| = -\ln\left(\frac{1}{2}x - a\right)$. [2]

[Questions 8 and 9 are printed overleaf.]

Jan 2007

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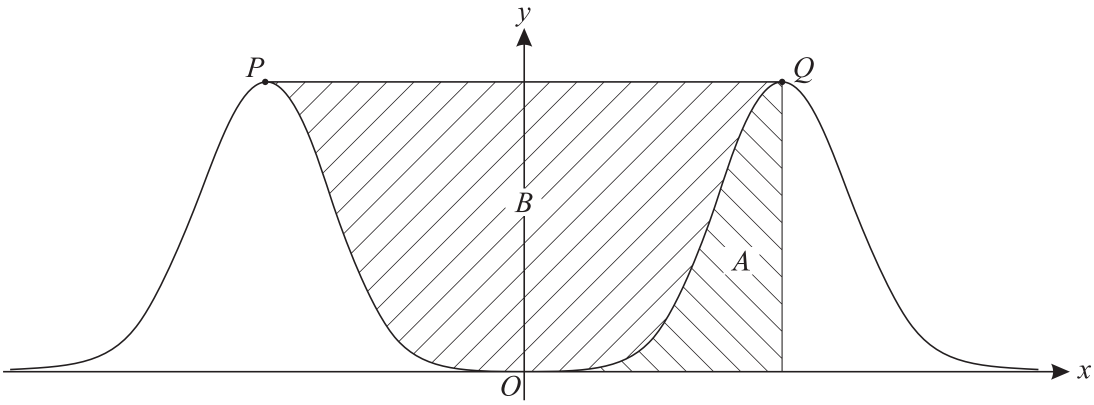


The diagram shows the curve with equation $y = x^8 e^{-x^2}$. The curve has maximum points at P and Q. The shaded region A is bounded by the curve, the line $y = 0$ and the line through Q parallel to the y-axis. The shaded region B is bounded by the curve and the line PQ.

(i) Show by differentiation that the x-coordinate of Q is 2. [5]

(ii) Use Simpson's rule with 4 strips to find an approximation to the area of region A. Give your answer correct to 3 decimal places. [4]

(iii) Deduce an approximation to the area of region B. [2]

9 Functions f and g are defined by

$$\mathrm{f}(x) = 2\sin x \quad \text{for } -\tfrac{1}{2}\pi \leqslant x \leqslant \tfrac{1}{2}\pi,$$
$$\mathrm{g}(x) = 4 - 2x^2 \quad \text{for } x \in \mathbb{R}.$$

(i) State the range of f and the range of g. [2]

(ii) Show that $\mathrm{gf}(0.5) = 2.16$, correct to 3 significant figures, and explain why $\mathrm{fg}(0.5)$ is not defined. [4]

(iii) Find the set of values of x for which $\mathrm{f}^{-1}\mathrm{g}(x)$ is not defined. [6]

June 2007

1 Differentiate each of the following with respect to x.

(i) $x^3(x+1)^5$ [2]

(ii) $\sqrt{3x^4+1}$ [3]

2 Solve the inequality $|4x-3| < |2x+1|$. [5]

3 The function f is defined for all non-negative values of x by

$$f(x) = 3 + \sqrt{x}.$$

(i) Evaluate ff(169). [2]

(ii) Find an expression for $f^{-1}(x)$ in terms of x. [2]

(iii) On a single diagram sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, indicating how the two graphs are related. [3]

4 The integral I is defined by

$$I = \int_0^{13} (2x+1)^{\frac{1}{3}}\,dx.$$

(i) Use integration to find the exact value of I. [4]

(ii) Use Simpson's rule with two strips to find an approximate value for I. Give your answer correct to 3 significant figures. [3]

5 A substance is decaying in such a way that its mass, m kg, at a time t years from now is given by the formula

$$m = 240e^{-0.04t}.$$

(i) Find the time taken for the substance to halve its mass. [3]

(ii) Find the value of t for which the mass is decreasing at a rate of 2.1 kg per year. [4]

6 **(i)** Given that $\int_0^a (6e^{2x} + x)\,dx = 42$, show that $a = \frac{1}{2}\ln\left(15 - \frac{1}{6}a^2\right)$. [5]

(ii) Use an iterative formula, based on the equation in part **(i)**, to find the value of a correct to 3 decimal places. Use a starting value of 1 and show the result of each iteration. [4]

June 2007

7 **(i)** Sketch the graph of $y = \sec x$ for $0 \leqslant x \leqslant 2\pi$. [2]

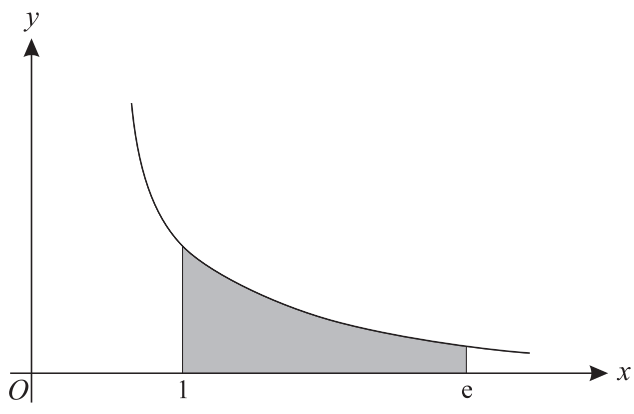
(ii) Solve the equation $\sec x = 3$ for $0 \leqslant x \leqslant 2\pi$, giving the roots correct to 3 significant figures. [3]

(iii) Solve the equation $\sec\theta = 5\operatorname{cosec}\theta$ for $0 \leqslant \theta \leqslant 2\pi$, giving the roots correct to 3 significant figures. [4]

8 **(i)** Given that $y = \dfrac{4\ln x - 3}{4\ln x + 3}$, show that $\dfrac{dy}{dx} = \dfrac{24}{x(4\ln x + 3)^2}$. [3]

(ii) Find the exact value of the gradient of the curve $y = \dfrac{4\ln x - 3}{4\ln x + 3}$ at the point where it crosses the x-axis. [4]

(iii)



The diagram shows part of the curve with equation

$$y = \frac{2}{x^{\frac{1}{2}}(4\ln x + 3)}.$$

The region shaded in the diagram is bounded by the curve and the lines $x = 1$, $x = \mathrm{e}$ and $y = 0$. Find the exact volume of the solid produced when this shaded region is rotated completely about the x-axis. [4]

9 **(i)** Prove the identity

$$\tan(\theta + 60^\circ)\tan(\theta - 60^\circ) \equiv \frac{\tan^2\theta - 3}{1 - 3\tan^2\theta}.$$ [4]

(ii) Solve, for $0^\circ < \theta < 180^\circ$, the equation

$$\tan(\theta + 60^\circ)\tan(\theta - 60^\circ) = 4\sec^2\theta - 3,$$

giving your answers correct to the nearest 0.1°. [5]

(iii) Show that, for all values of the constant k, the equation

$$\tan(\theta + 60^\circ)\tan(\theta - 60^\circ) = k^2$$

has two roots in the interval $0^\circ < \theta < 180^\circ$. [3]

Jan 2008

1 Functions f and g are defined for all real values of x by

$$f(x) = x^3 + 4 \quad \text{and} \quad g(x) = 2x - 5.$$

Evaluate

(i) fg(1), [2]

(ii) $f^{-1}(12)$. [3]

2 The sequence defined by

$$x_1 = 3, \qquad x_{n+1} = \sqrt[3]{31 - \tfrac{5}{2}x_n}$$

converges to the number α.

(i) Find the value of α correct to 3 decimal places, showing the result of each iteration. [3]

(ii) Find an equation of the form $ax^3 + bx + c = 0$, where a, b and c are integers, which has α as a root. [3]

3 **(a)** Solve, for $0^\circ < \alpha < 180^\circ$, the equation $\sec \tfrac{1}{2}\alpha = 4$. [3]

(b) Solve, for $0^\circ < \beta < 180^\circ$, the equation $\tan \beta = 7 \cot \beta$. [4]

4 Earth is being added to a pile so that, when the height of the pile is h metres, its volume is V cubic metres, where

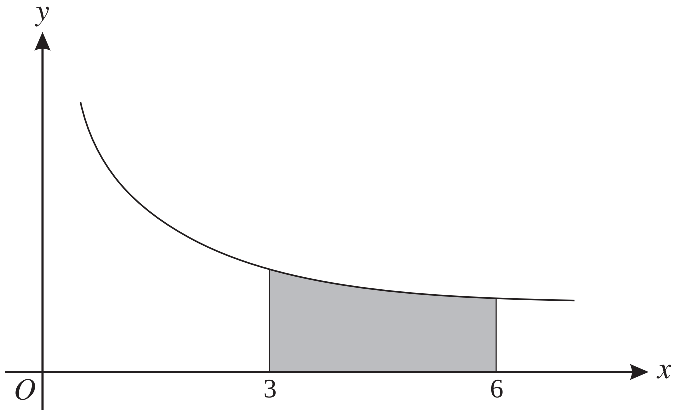
$$V = (h^6 + 16)^{\frac{1}{2}} - 4.$$

(i) Find the value of $\dfrac{dV}{dh}$ when $h = 2$. [3]

(ii) The volume of the pile is increasing at a constant rate of 8 cubic metres per hour. Find the rate, in metres per hour, at which the height of the pile is increasing at the instant when $h = 2$. Give your answer correct to 2 significant figures. [3]

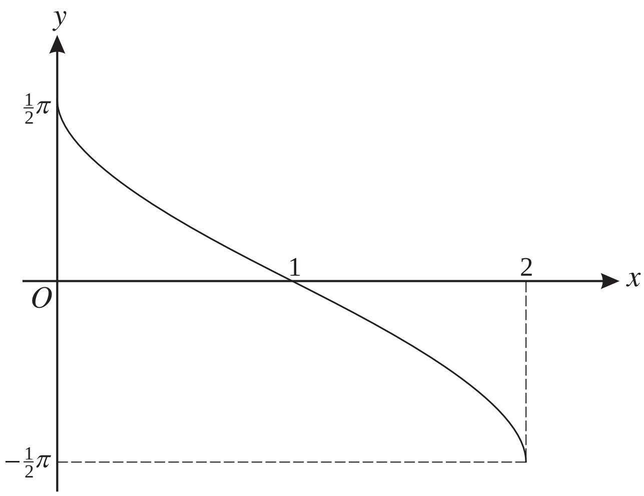
5 **(a)** Find $\int (3x+7)^9 \, dx$. [3]

(b)



The diagram shows the curve $y = \frac{1}{2\sqrt{x}}$. The shaded region is bounded by the curve and the lines $x = 3$, $x = 6$ and $y = 0$. The shaded region is rotated completely about the x-axis. Find the exact volume of the solid produced, simplifying your answer. [5]

6



The diagram shows the graph of $y = -\sin^{-1}(x-1)$.

(i) Give details of the pair of geometrical transformations which transforms the graph of $y = -\sin^{-1}(x-1)$ to the graph of $y = \sin^{-1} x$. [3]

(ii) Sketch the graph of $y = |-\sin^{-1}(x-1)|$. [2]

(iii) Find the exact solutions of the equation $|-\sin^{-1}(x-1)| = \frac{1}{3}\pi$. [3]

Jan 2008

7 A curve has equation $y = \frac{xe^{2x}}{x+k}$, where k is a non-zero constant.

(i) Differentiate xe^{2x}, and show that $\frac{dy}{dx} = \frac{e^{2x}(2x^2 + 2kx + k)}{(x+k)^2}$. [5]

(ii) Given that the curve has exactly one stationary point, find the value of k, and determine the exact coordinates of the stationary point. [5]

8 The definite integral I is defined by

$$I = \int_0^6 2^x \, dx.$$

(i) Use Simpson's rule with 6 strips to find an approximate value of I. [4]

(ii) By first writing 2^x in the form e^{kx}, where the constant k is to be determined, find the exact value of I. [4]

(iii) Use the answers to parts **(i)** and **(ii)** to deduce that $\ln 2 \approx \frac{9}{13}$. [2]

9 **(i)** Use the identity for $\cos(A + B)$ to prove that

$$4\cos(\theta + 60^\circ)\cos(\theta + 30^\circ) \equiv \sqrt{3} - 2\sin 2\theta. \quad [4]$$

(ii) Hence find the exact value of $4\cos 82.5^\circ \cos 52.5^\circ$. [2]

(iii) Solve, for $0^\circ < \theta < 90^\circ$, the equation $4\cos(\theta + 60^\circ)\cos(\theta + 30^\circ) = 1$. [3]

(iv) Given that there are no values of θ which satisfy the equation

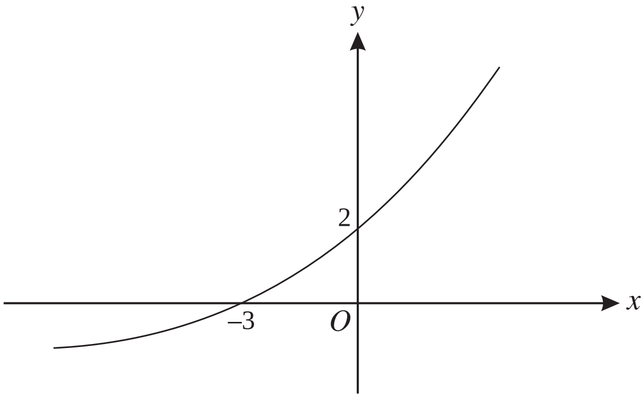
$$4\cos(\theta + 60^\circ)\cos(\theta + 30^\circ) = k,$$

determine the set of values of the constant k. [3]

June 2008

1 Find the exact solutions of the equation $|4x - 5| = |3x - 5|$. [4]

2



The diagram shows the graph of $y = \mathrm{f}(x)$. It is given that $\mathrm{f}(-3) = 0$ and $\mathrm{f}(0) = 2$. Sketch, on separate diagrams, the following graphs, indicating in each case the coordinates of the points where the graph crosses the axes:

(i) $y = \mathrm{f}^{-1}(x)$, [2]

(ii) $y = -2\mathrm{f}(x)$. [3]

3 Find, in the form $y = mx + c$, the equation of the tangent to the curve

$$y = x^2 \ln x$$

at the point with x-coordinate e. [6]

4 The gradient of the curve $y = (2x^2 + 9)^{\frac{5}{2}}$ at the point P is 100.

(i) Show that the x-coordinate of P satisfies the equation $x = 10(2x^2 + 9)^{-\frac{3}{2}}$. [3]

(ii) Show by calculation that the x-coordinate of P lies between 0.3 and 0.4. [3]

(iii) Use an iterative formula, based on the equation in part **(i)**, to find the x-coordinate of P correct to 4 decimal places. You should show the result of each iteration. [3]

5 **(a)** Express $\tan 2\alpha$ in terms of $\tan\alpha$ and hence solve, for $0^\circ < \alpha < 180^\circ$, the equation

$$\tan 2\alpha \tan\alpha = 8.$$ [6]

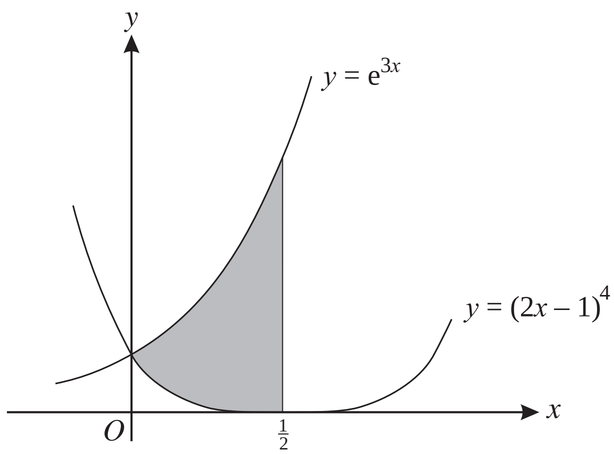
(b) Given that β is the acute angle such that $\sin\beta = \frac{6}{7}$, find the exact value of

(i) $\operatorname{cosec}\beta$, [1]

(ii) $\cot^2\beta$. [2]

June 2008

6



The diagram shows the curves $y = e^{3x}$ and $y = (2x - 1)^4$. The shaded region is bounded by the two curves and the line $x = \frac{1}{2}$. The shaded region is rotated completely about the x-axis. Find the exact volume of the solid produced. [9]

7 It is claimed that the number of plants of a certain species in a particular locality is doubling every 9 years. The number of plants now is 42. The number of plants is treated as a continuous variable and is denoted by N. The number of years from now is denoted by t.

(i) Two equivalent expressions giving N in terms of t are

$$N = A \times 2^{kt} \qquad \text{and} \qquad N = Ae^{mt}.$$

Determine the value of each of the constants A, k and m. [4]

(ii) Find the value of t for which $N = 100$, giving your answer correct to 3 significant figures. [2]

(iii) Find the rate at which the number of plants will be increasing at a time 35 years from now. [3]

8 The expression $\mathrm{T}(\theta)$ is defined for θ in degrees by

$$\mathrm{T}(\theta) = 3\cos(\theta - 60^\circ) + 2\cos(\theta + 60^\circ).$$

(i) Express $\mathrm{T}(\theta)$ in the form $A\sin\theta + B\cos\theta$, giving the exact values of the constants A and B. [3]

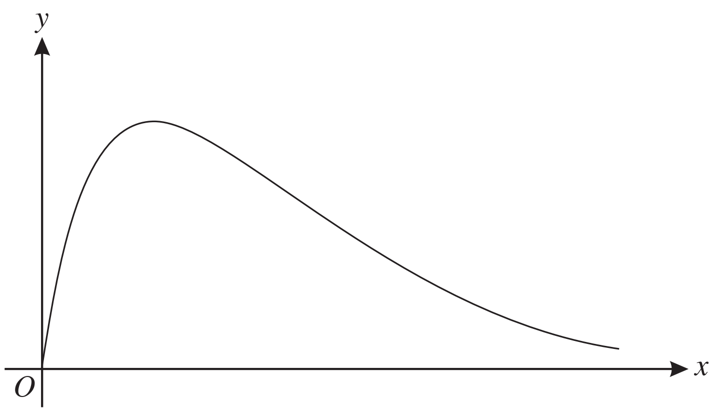
(ii) Hence express $\mathrm{T}(\theta)$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. [3]

(iii) Find the smallest positive value of θ such that $\mathrm{T}(\theta) + 1 = 0$. [4]

[Question 9 is printed overleaf.]

June 2008

9



The function f is defined for the domain $x \geqslant 0$ by

$$f(x) = \frac{15x}{x^2 + 5}.$$

The diagram shows the curve with equation $y = f(x)$.

(i) Find the range of f. [6]

(ii) The function g is defined for the domain $x \geqslant k$ by

$$g(x) = \frac{15x}{x^2 + 5}.$$

Given that g is a one-one function, state the least possible value of k. [1]

(iii) Show that there is no point on the curve $y = g(x)$ at which the gradient is -1. [4]

Jan 2009

1 Find

(i) $\int 8e^{-2x}\,dx$,

(ii) $\int (4x+5)^6\,dx$. **[5]**

2 **(i)** Use Simpson's rule with four strips to find an approximation to

$$\int_4^{12} \ln x\,dx,$$

giving your answer correct to 2 decimal places. **[4]**

(ii) Deduce an approximation to $\int_4^{12} \ln(x^{10})\,dx$. **[1]**

3 **(i)** Express $2\tan^2\theta - \frac{1}{\cos\theta}$ in terms of $\sec\theta$. **[3]**

(ii) Hence solve, for $0° < \theta < 360°$, the equation

$$2\tan^2\theta - \frac{1}{\cos\theta} = 4.$$ **[4]**

4 For each of the following curves, find $\frac{dy}{dx}$ and determine the exact x-coordinate of the stationary point:

(i) $y = (4x^2+1)^5$, **[3]**

(ii) $y = \frac{x^2}{\ln x}$. **[4]**

5 The mass, M grams, of a certain substance is increasing exponentially so that, at time t hours, the mass is given by

$$M = 40e^{kt},$$

where k is a constant. The following table shows certain values of t and M.

t	0	21	63
M		80	

(i) In either order,

(a) find the values missing from the table, **[3]**

(b) determine the value of k. **[2]**

(ii) Find the rate at which the mass is increasing when $t = 21$. **[3]**

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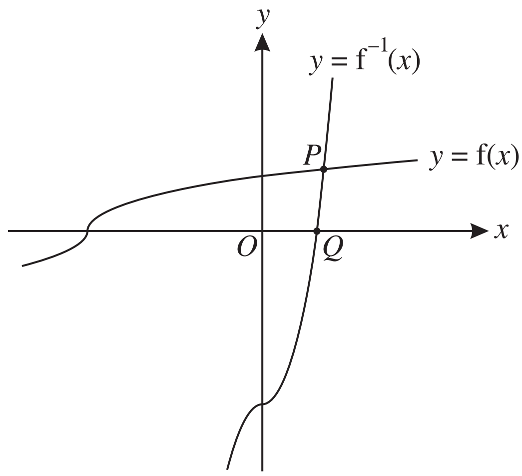
6



The function f is defined for all real values of x by

$$f(x) = \sqrt[3]{\tfrac{1}{2}x + 2}.$$

The graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet at the point P, and the graph of $y = f^{-1}(x)$ meets the x-axis at Q (see diagram).

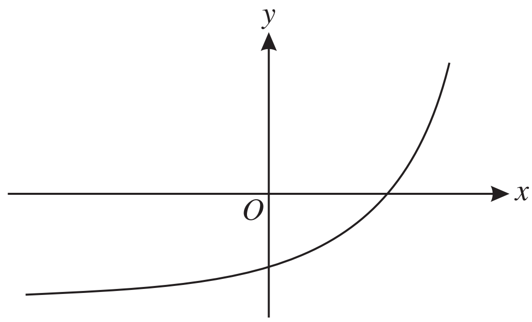
(i) Find an expression for $f^{-1}(x)$ and determine the x-coordinate of the point Q. **[3]**

(ii) State how the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are related geometrically, and hence show that the x-coordinate of the point P is the root of the equation

$$x = \sqrt[3]{\tfrac{1}{2}x + 2}.$$ **[2]**

(iii) Use an iterative process, based on the equation $x = \sqrt[3]{\tfrac{1}{2}x + 2}$, to find the x-coordinate of P, giving your answer correct to 2 decimal places. **[4]**

7



The diagram shows the curve $y = e^{kx} - a$, where k and a are constants.

(i) Give details of the pair of transformations which transforms the curve $y = e^x$ to the curve $y = e^{kx} - a$. **[3]**

(ii) Sketch the curve $y = |e^{kx} - a|$. **[2]**

(iii) Given that the curve $y = |e^{kx} - a|$ passes through the points $(0, 13)$ and $(\ln 3, 13)$, find the values of k and a. **[4]**

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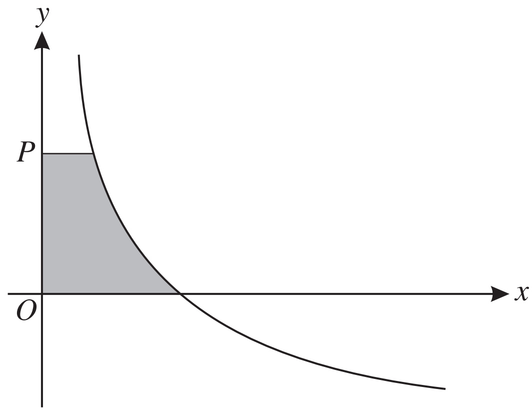
8



The diagram shows the curve with equation

$$y = \frac{6}{\sqrt{x}} - 3.$$

The point P has coordinates $(0, p)$. The shaded region is bounded by the curve and the lines $x = 0$, $y = 0$ and $y = p$. The shaded region is rotated completely about the y-axis to form a solid of volume V.

(i) Show that $V = 16\pi\left(1 - \frac{27}{(p+3)^3}\right)$. **[6]**

(ii) It is given that P is moving along the y-axis in such a way that, at time t, the variables p and t are related by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \tfrac{1}{3}p + 1.$$

Find the value of $\frac{\mathrm{d}V}{\mathrm{d}t}$ at the instant when $p = 9$. **[4]**

9 **(i)** By first expanding $\cos(2\theta + \theta)$, prove that

$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta. \quad \textbf{[4]}$$

(ii) Hence prove that

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1. \quad \textbf{[3]}$$

(iii) Show that the only solutions of the equation

$$1 + \cos 6\theta = 18\cos^2\theta$$

are odd multiples of 90°. **[5]**

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1

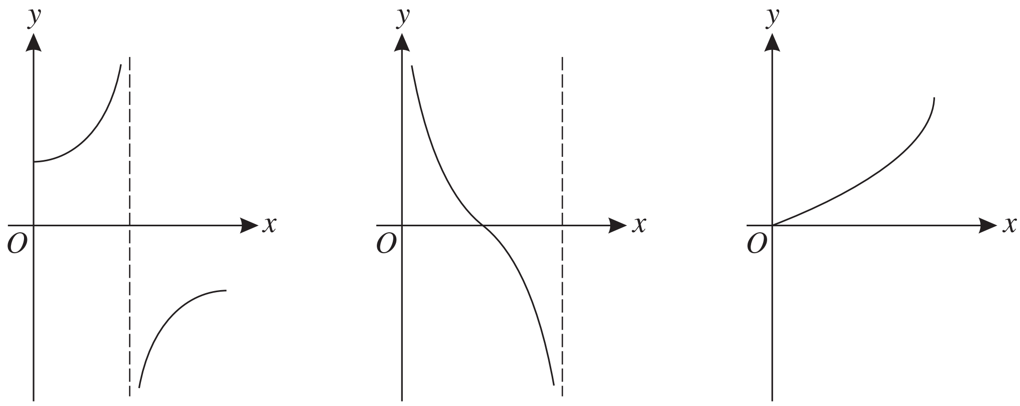


Fig. 1 **Fig. 2** **Fig. 3**

Each diagram above shows part of a curve, the equation of which is one of the following:

$$y = \sin^{-1} x, \quad y = \cos^{-1} x, \quad y = \tan^{-1} x, \quad y = \sec x, \quad y = \operatorname{cosec} x, \quad y = \cot x.$$

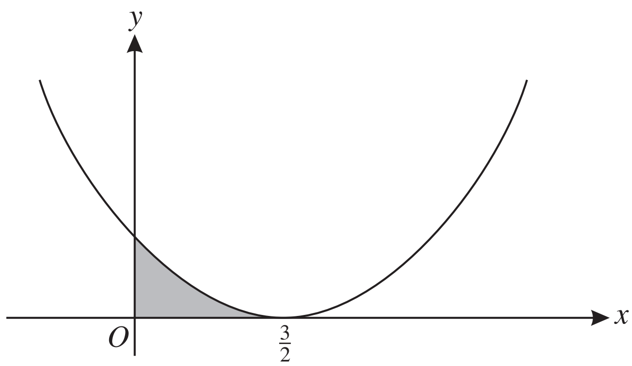
State which equation corresponds to

(i) Fig. 1, **[1]**

(ii) Fig. 2, **[1]**

(iii) Fig. 3. **[1]**

2



The diagram shows the curve with equation $y = (2x - 3)^2$. The shaded region is bounded by the curve and the lines $x = 0$ and $y = 0$. Find the exact volume obtained when the shaded region is rotated completely about the x-axis. **[5]**

3 The angles α and β are such that

$$\tan\alpha = m + 2 \quad \text{and} \quad \tan\beta = m,$$

where m is a constant.

(i) Given that $\sec^2\alpha - \sec^2\beta = 16$, find the value of m. **[3]**

(ii) Hence find the exact value of $\tan(\alpha + \beta)$. **[3]**

June 2009

4 It is given that $\int_a^{3a} (e^{3x} + e^x)\,dx = 100$, where a is a positive constant.

(i) Show that $a = \frac{1}{9}\ln(300 + 3e^a - 2e^{3a})$. **[5]**

(ii) Use an iterative process, based on the equation in part **(i)**, to find the value of a correct to 4 decimal places. Use a starting value of 0.6 and show the result of each step of the process. **[4]**

5 The functions f and g are defined for all real values of x by

$$f(x) = 3x - 2 \quad \text{and} \quad g(x) = 3x + 7.$$

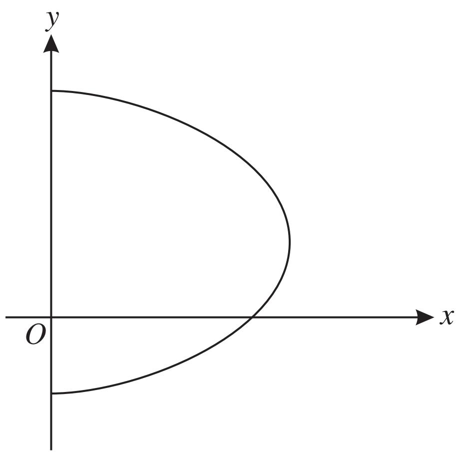
Find the exact coordinates of the point at which

(i) the graph of $y = fg(x)$ meets the x-axis, **[3]**

(ii) the graph of $y = g(x)$ meets the graph of $y = g^{-1}(x)$, **[3]**

(iii) the graph of $y = |f(x)|$ meets the graph of $y = |g(x)|$. **[4]**

6



The diagram shows the curve with equation $x = (37 + 10y - 2y^2)^{\frac{1}{2}}$.

(i) Find an expression for $\frac{dx}{dy}$ in terms of y. **[2]**

(ii) Hence find the equation of the tangent to the curve at the point (7, 3), giving your answer in the form $y = mx + c$. **[5]**

7 **(i)** Express $8\sin\theta - 6\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. **[3]**

(ii) Hence

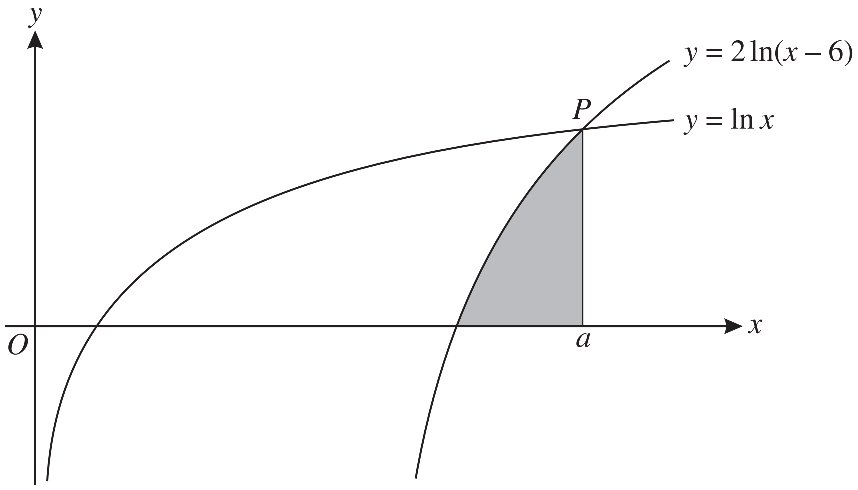
(a) solve, for $0° < \theta < 360°$, the equation $8\sin\theta - 6\cos\theta = 9$, **[4]**

(b) find the greatest possible value of

$$32\sin x - 24\cos x - (16\sin y - 12\cos y)$$

as the angles x and y vary. **[3]**

8



The diagram shows the curves $y = \ln x$ and $y = 2\ln(x - 6)$. The curves meet at the point P which has x-coordinate a. The shaded region is bounded by the curve $y = 2\ln(x - 6)$ and the lines $x = a$ and $y = 0$.

(i) Give details of the pair of transformations which transforms the curve $y = \ln x$ to the curve $y = 2\ln(x - 6)$. **[3]**

(ii) Solve an equation to find the value of a. **[4]**

(iii) Use Simpson's rule with two strips to find an approximation to the area of the shaded region. **[3]**

9 **(a)** Show that, for all non-zero values of the constant k, the curve

$$y = \frac{kx^2 - 1}{kx^2 + 1}$$

has exactly one stationary point. **[5]**

(b) Show that, for all non-zero values of the constant m, the curve

$$y = e^{mx}(x^2 + mx)$$

has exactly two stationary points. **[7]**